

# EXACT BOUNDARY CONTROLLABILITY RESULTS FOR A MULTILAYER RAO-NAKRA SANDWICH BEAM

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**Abstract.** We study the boundary controllability problem for a multilayer Rao-Nakra sandwich beam. This beam model consists of a Rayleigh beam coupled with a number of wave equations. We consider all combinations of clamped and hinged boundary conditions with the control applied to either the moment or the rotation angle at an end of the beam. We prove that exact controllability holds provided the damping parameter is sufficiently small. In the undamped case, exact controllability holds without any restriction on the parameters in the system. In each case, optimal control time is obtained in the space of optimal regularity for  $L^2(0, T)$  controls. A key step in the proof of our main result is the proof of uniqueness of the zero solution of the eigensystem with the homogeneous boundary conditions together with zero boundary observation.

**Key words.** Boundary control, exact controllability, multiplier method, multilayer beam, sandwich beam, Rayleigh beam.

**1. Introduction.** The classical sandwich beam is an engineering model for a three layer beam consisting of two “face plates” and a “core” layer that is orders of magnitude more compliant than the face plates. While most of the early models considered only transverse dynamics, e.g., [12], [20], the model due to Rao and Nakra [17] includes rotary inertia in each layer and longitudinal inertia (in addition to transverse inertia). The model assumes continuous, piecewise linear displacements through the cross-sections, with the Kirchhoff hypothesis imposed on the face plates.

In this article we study the boundary controllability of the following multilayer generalization of the Rao-Nakra beam derived in [1]:

$$\begin{cases} m\ddot{w} - \alpha\ddot{w}'' + Kw'''' - N^T \mathbf{h}_E (\mathbf{G}_E \psi_E + \tilde{\mathbf{G}}_E \dot{\psi}_E)' = 0 & \text{in } \Omega \times \mathbb{R}^+ \\ \mathbf{h}_O \mathbf{p}_O \ddot{y}_O - \mathbf{h}_O \mathbf{E}_O y_O'' + \mathbf{B}^T (\mathbf{G}_E \psi_E + \tilde{\mathbf{G}}_E \dot{\psi}_E) = 0 & \text{on } \Omega \times \mathbb{R}^+ \\ \text{where } \mathbf{B} y_O = \mathbf{h}_E \psi_E - \mathbf{h}_E N w', \end{cases} \quad (1.1)$$

where  $\Omega = (0, L)$ , primes denote differentiation with respect to the spatial variable  $x$  and dots denote differentiation with respect to time  $t$ .

The model (1.1) consists of  $2m + 1$  alternating stiff and compliant (core) layers, with stiff layers on outside. The stiff layers have odd indices  $1, 3, \dots, 2m + 1$  and the even layers have even indices  $2, 4, \dots, 2m$ . The Kirchhoff hypothesis is imposed on the stiff layers and Timoshenko displacement assumptions are assumed in the compliant layers. Damping proportional rate of shear is included in the compliant layers.

In the above,  $m, \alpha, K$  are *positive* physical constants,  $w$  represents the transverse displacement,  $\psi^i$  denotes the shear angle in the  $i^{\text{th}}$  layer,  $\psi_E = [\psi^2, \psi^4, \dots, \psi^{2m}]^T$ ,  $y^i$  denote the longitudinal displacement along the center of the  $i^{\text{th}}$  layer, and  $y_O = [y^1, y^3, \dots, y^{2m+1}]^T$ , and

$$\mathbf{p}_O = \text{diag}(\rho_1, \dots, \rho_{2m+1}), \quad \mathbf{h}_O = \text{diag}(h_1, \dots, h_{2m+1}), \quad \mathbf{h}_E = \text{diag}(h_2, \dots, h_{2m}), \\ \mathbf{E}_O = \text{diag}(E_1, \dots, E_{2m+1}), \quad \mathbf{G}_E = \text{diag}(G_2, \dots, G_{2m}), \quad \tilde{\mathbf{G}}_E = \text{diag}(\tilde{G}_2, \dots, \tilde{G}_{2m})$$

where  $h_i, \rho_i, E_i$ , are positive and denote the thickness, density, and Young's modulus, respectively. Also  $G_i \geq 0$  denotes shear modulus of the  $i^{\text{th}}$  layer, and  $\tilde{G}_i \geq 0$  denotes coefficient for damping in the corresponding compliant layer.

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The vector  $N$  is defined as  $N = \mathbf{h}_E^{-1} \mathbf{A} \mathbf{h}_O \vec{1}_O + \vec{1}_E$  where  $\mathbf{A} = (a_{ij})$  and  $\mathbf{B} = (b_{ij})$  are the  $m \times (m+1)$  matrices

$$a_{ij} = \begin{cases} 1/2, & \text{if } j = i \text{ or } j = i+1 \\ 0, & \text{otherwise} \end{cases}, \quad b_{ij} = \begin{cases} (-1)^{i+j+1}, & \text{if } j = i \text{ or } j = i+1 \\ 0, & \text{otherwise} \end{cases}$$

and  $\vec{1}_O$  and  $\vec{1}_E$  denote the vectors with all entries 1 in  $\mathbb{R}^{m+1}$  and  $\mathbb{R}^m$ , respectively.

Consider (1.1) with either hinged-Neumann (h-N), or clamped-Dirichlet (c-D), or mixed-mixed (m-m) boundary conditions respectively

$$\left\{ \begin{array}{l} w(0, t) = w''(0, t) = w(L, t) = 0, w''(L, t) = M(t) \text{ on } \mathbb{R}^+ \\ y'_O(0, t) = 0, y'_O(L, t) = \mathbf{g}_O(t) \text{ on } \mathbb{R}^+, \end{array} \right\} \quad (\text{h-N}) \quad (1.2)$$

$$\left\{ \begin{array}{l} w(0, t) = w'(0, t) = w(L, t) = 0, w'(L, t) = M(t) \text{ on } \mathbb{R}^+ \\ y_O(0, t) = 0, y_O(L, t) = \mathbf{g}_O(t) \text{ on } \mathbb{R}^+, \end{array} \right\} \quad (\text{c-D}) \quad (1.3)$$

$$\left\{ \begin{array}{l} w(0, t) = w'(0, t) = w(L, t) = 0, w''(L, t) = M(t) \text{ on } \mathbb{R}^+ \\ y_O(0, t) = 0, y'_O(L, t) = \mathbf{g}_O(t) \text{ on } \mathbb{R}^+. \end{array} \right\} \quad (\text{m-m}) \quad (1.4)$$

The initial conditions for (1.1) are

$$w(x, 0) = w^0(x), \quad \dot{w}(x, 0) = w^1(x), \quad y_O(x, 0) = y_O^0, \quad \dot{y}_O(x, 0) = y_O^1 \text{ on } \Omega. \quad (1.5)$$

In this paper, through the controls  $M(t)$  and  $\mathbf{g}_O(t)$  at the right end of the beam, we control the moment and longitudinal force of the stiff layers in (1.2) and (1.4), and the shear angle and the longitudinal displacements of the stiff layers in (1.3).

**1.1. Background.** In [16], exact boundary controllability of three-layer Rao-Nakra beam was investigated for the boundary conditions (1.3). An exact controllability result for sufficiently large control time but with size restrictions on the coupling parameters ( $\tilde{\mathbf{G}}$  and  $\mathbf{G}$  in (1.1)) was obtained by the standard multiplier method. In [4], the moment method was applied to the three-layer Rao-Nakra system with the boundary conditions (1.2). Under the assumption of distinct wave speeds, exact controllability was shown up to a finite-dimensional subspace which consists of low-frequency eigenvectors of the system. With additional restrictions on the parameters ( $\tilde{\mathbf{G}}$  and  $\mathbf{G}$  in (1.1)), and exact controllability of the vibrational states was obtained. Exponential boundary feedback stabilization results for a related (but different) three layer laminated beam were obtained in [18]. In [2], [3] exact controllability results for the multilayer Rao-Nakra plate system analogous to (1.1) with locally distributed control in a neighborhood of a portion of the boundary were obtained by the method of Carleman estimates.

**1.2. Main results.** Let

$$\mathcal{C} = \begin{cases} (H^2(\Omega) \cap H_0^1(\Omega)) \times (\tilde{H}^1(\Omega))^{(m+1)} \times H_0^1(\Omega) \times (\tilde{L}^2(\Omega))^{(m+1)} & (\text{h-N}) \quad (1.6a) \\ H_0^1(\Omega) \times (L^2(\Omega))^{(m+1)} \times (L^2(\Omega)/\mathbb{M}) \times (H^{-1}(\Omega))^{(m+1)} & (\text{c-D}) \quad (1.6b) \\ H_{\#}^2(\Omega) \times (H_{\dagger}^1(\Omega))^{(m+1)} \times H_0^1(\Omega) \times (L^2(\Omega))^{(m+1)} & (\text{m-m}) \quad (1.6c) \end{cases}$$

where  $\tilde{H}^1(\Omega)$  and  $\tilde{L}^2(\Omega)$  are the quotient spaces defined by  $\tilde{H}^1(\Omega) = H^1(\Omega)/\mathbb{R}$  and  $\tilde{L}^2(\Omega) = L^2(\Omega)/\mathbb{R}$  respectively, and

$$\begin{aligned} \mathbb{M} &= \text{span}\{e^{-\frac{1}{\sqrt{\alpha/m}}x}, e^{\frac{1}{\sqrt{\alpha/m}}x}\}, \\ H_{\#}^2(\Omega) &= \{u \in H^2(\Omega) \cap H_0^1(\Omega) : u'(0) = 0\}, \\ H_{\dagger}^1(\Omega) &= \{u \in H^1(\Omega) : u(0) = 0\}. \end{aligned} \quad (1.7)$$

PROPOSITION 1.1. *Let  $T > 0$ , and  $(M(t), \mathbf{g}_\mathcal{O}(t)) \in (L^2(0, T))^{(m+2)}$ . For any  $(w^0, y_\mathcal{O}^0, w^1, y_\mathcal{O}^1)^\mathsf{T} \in \mathcal{C}$ , there exists a unique solution  $(w, y_\mathcal{O}, \dot{w}, \dot{y}_\mathcal{O})^\mathsf{T}$  to (1.1)-(1.5) with  $(w, y_\mathcal{O}, \dot{w}, \dot{y}_\mathcal{O})^\mathsf{T} \in C([0, T]; \mathcal{C})$  and*

$$\|(w, y_\mathcal{O}, \dot{w}, \dot{y}_\mathcal{O})^\mathsf{T}\|_\mathcal{C} \leq C \left\{ \|(w^0, y_\mathcal{O}^0, w^1, y_\mathcal{O}^1)^\mathsf{T}\|_\mathcal{C} + \|(M, \mathbf{g}_\mathcal{O})\|_{(L^2(\Omega))^{(m+2)}} \right\}.$$

Our main exact controllability theorem is the following:

THEOREM 1.1. *Let  $T > \tau$  where*

$$\tau := 2L \left[ \min_{i=1,3,\dots,2m+1} \left( \sqrt{\frac{K}{\alpha}}, \sqrt{\frac{\rho_i}{E_i}} \right) \right]^{-1}. \quad (1.8)$$

*For sufficiently small  $\|\tilde{\mathbf{G}}_E\|$  and for any  $(w^0, y_\mathcal{O}^0, w^1, y_\mathcal{O}^1)^\mathsf{T} \in \mathcal{C}$  there exists  $(M(t), \mathbf{g}_\mathcal{O}(t)) \in (L^2(0, T))^{(m+2)}$  such that  $(w(T), y_\mathcal{O}(T), \dot{w}(T), \dot{y}_\mathcal{O}(T))^\mathsf{T} = 0$ .*

Now consider

$$\begin{cases} m\ddot{z} - \alpha z'' + K z'''' - N^\mathsf{T} \mathbf{h}_E \left( \mathbf{G}_E \phi_E + \tilde{\mathbf{G}}_E \dot{\phi}_E \right)' = 0 & \text{on } \Omega \times \mathbb{R}^+ \\ \mathbf{h}_\mathcal{O} \mathbf{p}_\mathcal{O} \ddot{v}_\mathcal{O} - \mathbf{h}_\mathcal{O} \mathbf{E}_\mathcal{O} v_\mathcal{O}'' + \mathbf{B}^\mathsf{T} \left( \mathbf{G}_E \phi_E + \tilde{\mathbf{G}}_E \dot{\phi}_E \right) = 0 & \text{on } \Omega \times \mathbb{R}^+ \\ \text{where } \mathbf{B} v_\mathcal{O} = \mathbf{h}_E \phi_E - \mathbf{h}_E N z' \end{cases} \quad (1.9)$$

with either hinged-Neumann (h-N), or clamped-Dirichlet (c-D), or mixed-mixed (m-m) boundary conditions respectively

$$\begin{cases} z(0, t) = z''(0, t) = z(L, t) = z''(L, t) = 0, v_\mathcal{O}'(0, t) = v_\mathcal{O}'(L, t) = 0 & \text{(h-N)} \quad (1.10a) \\ z(0, t) = z'(0, t) = z(L, t) = z'(L, t) = 0, v_\mathcal{O}(0, t) = v_\mathcal{O}(L, t) = 0 & \text{(c-D)} \quad (1.10b) \\ z(0, t) = z'(0, t) = z(L, t) = z''(L, t) = 0, v_\mathcal{O}(0, t) = v_\mathcal{O}'(L, t) = 0. & \text{(m-m)} \quad (1.10c) \end{cases}$$

The initial conditions for (1.9) are

$$z(x, 0) = z^0(x), \quad \dot{z}(x, 0) = z^1(x), \quad v_\mathcal{O}(x, 0) = v_\mathcal{O}^0, \quad \dot{v}_\mathcal{O}(x, 0) = v_\mathcal{O}^1. \quad (1.11)$$

For convenience, let  $\mathcal{S}$  be a set, and  $f, g$  be nonnegative functions on  $\mathcal{S}$ . We will write  $f \asymp g$  if there exists  $C > 0$  such that

$$\frac{1}{C} f(\lambda) \leq g(\lambda) \leq C f(\lambda), \quad \forall \lambda \in \mathcal{S}.$$

The results in Theorem 1.1 are based upon the following observability and hidden regularity results:

THEOREM 1.2. *Let  $T > \tau$ . Then for sufficiently small  $\|\tilde{\mathbf{G}}_E\|$  solutions of the problem (1.9)- (1.11) satisfy the following observability and hidden regularity estimates:*

$$\begin{cases} \int_0^T (|z'''(L, t)|^2 + |v_\mathcal{O}''(L, t)|^2) dt \asymp \|(z^0, v_\mathcal{O}^0, z^1, v_\mathcal{O}^1)^\mathsf{T}\|_\mathcal{H}^2 & \text{(h-N)} \quad (1.12a) \\ \int_0^T (|z''(L, t)|^2 + |v_\mathcal{O}'(L, t)|^2) dt \asymp \|(z^0, v_\mathcal{O}^0, z^1, v_\mathcal{O}^1)^\mathsf{T}\|_\mathcal{H}^2 & \text{(c-D)} \quad (1.12b) \\ \int_0^T (|z'(L, t)|^2 + |v_\mathcal{O}(L, t)|^2) dt \asymp \|(z^0, v_\mathcal{O}^0, z^1, v_\mathcal{O}^1)^\mathsf{T}\|_{\mathcal{H}_{-1}}^2 & \text{(m-m)} \quad (1.12c) \end{cases}$$

where  $\mathcal{H}$  and  $\mathcal{H}_{-1}$  are later defined in (2.6) and (3.10), respectively.

Our results are improvements on earlier results [4], [16] in several regards. Here, we consider the general multilayer system. The restriction on the size of  $\mathbf{G}$  has been eliminated, there are no conditions on the wave speeds, and the optimal control time (determined by characteristics) is obtained.

Our overall methodology is to first obtain appropriate boundary observability estimates for the uncoupled system of equations. This part uses mainly known estimates for the wave equation together with observability results obtained in [14]. Second, we prove, based on carefully picked complex multipliers, a uniqueness result (Lemma 4.1) for the over-determined eigensystem of the coupled system without damping  $\mathbf{G} = 0$  consisting of the homogeneous boundary conditions together with zero observation. This allows us to deduce (using Theorem 6.2 in [6]) observability of the coupled system without damping. Finally, we are able to include the possibility of small damping by a perturbation argument.

We consider three different sets of boundary conditions. While the overall structure of the proofs are the same in each case, the spaces that arise are different and lead to some very different technical issues. For example, in the case of (h-N) boundary conditions, the system is well-posed with respect to a higher-order energy defined by an extra derivative applied to each variable. This allows us to obtain (similar to [5], [7], [8]) an observability result in a correspondingly smooth space, which is equivalent to controllability in the natural energy space. This approach fails in the case of (m-m) boundary conditions, where instead, we obtain an observability result for weaker solutions in which certain orthogonality conditions arise (see Lemma 3.1). In the case of (c-D) boundary conditions we obtain an observability result in the standard energy space, which in turn corresponds to an exact controllability result in a weaker space involving a quotient  $M$  in the velocity component of the transverse displacement in (1.1). The quotient  $M$  can not be eliminated if  $L^2(0, T)$  controls are used. This is due to orthogonality conditions on the range of the operator  $\mathcal{L}\phi = m\phi - \alpha\phi''$  on the domain  $H_0^2(\Omega)$  which must be imposed in the transpositional solution. (See Section 5.2 for details.) In fact, a quotient space analogous to  $M$  was found in the velocity component of the optimal controls for boundary control of the Kirchhoff plate with clamped boundary conditions, [9]. Related optimal controllability and observability results for the Rayleigh beam are described in [14].

All of the controllability results in this paper are optimal in the sense that the space of exact controllability matches the optimal regularity space for  $L^2(0, T)$  boundary controls. Moreover, as mentioned above, the quotient  $M$  in (1.6b) can not be eliminated from the control space if  $L^2(0, T)$  controls are used. On the other hand, the quotients that occur in the second and fourth components of the control space (1.6a) are perhaps inessential in that they arise as a consequence of orthogonality constraints imposed for convenience in the homogeneous solutions (see (2.6a)) which are used in the definition of transpositional solution (see Definition 5.1). In this case solutions in (1.6a) are defined up to uniform translational motion in each layer.

This paper is organized as follows. In Section 2 we prove regularity results for the homogeneous system using semigroup theory. In Section 3 we characterize the weaker observability space for the case of (m-m) boundary conditions. In Section 4 we prove the key uniqueness result Lemma 4.1 and main observability result Theorem 1.2. In Section 5 we define transpositional solutions of the control problem and prove our main controllability result Theorem 1.1.

## 2. Semigroup formulation. Let

$$U =: (u, \mathbf{u})^T = (z, v_{\mathcal{O}})^T, \quad V =: (v, \mathbf{v})^T = (\dot{z}, \dot{v}_{\mathcal{O}})^T, \quad \text{and} \quad Y := (U, V)^T.$$

Let  $\mathcal{L}\varphi = m\varphi - \alpha\varphi''$ . From the Lax-Milgram theorem  $\mathcal{L} : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$  is an isomorphism which remains isomorphic from  $H^2(\Omega) \cap H_0^1(\Omega)$  to  $L^2(\Omega)$ .

Then (1.9)-(1.11) can be written as

$$\frac{dY}{dt} = \mathcal{A}Y := \begin{pmatrix} 0 & I \\ -A_1 & A_2 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}, \quad Y(0) = (U(0), V(0))^T = (z^0, v_{\mathcal{O}}^0, z^1, v_{\mathcal{O}}^1)^T \quad (2.1)$$

where

$$A_1 U := \begin{pmatrix} \mathcal{L}^{-1} (K u'''' - N^T \mathbf{h}_E \mathbf{G}_E (\mathbf{h}_E^{-1} \mathbf{B} \mathbf{u}' + N u'')) \\ \mathbf{h}_{\mathcal{O}}^{-1} \mathbf{p}_{\mathcal{O}}^{-1} (-\mathbf{h}_{\mathcal{O}} \mathbf{E}_{\mathcal{O}} \mathbf{u}'' + \mathbf{B}^T \mathbf{G}_E (\mathbf{h}_E^{-1} \mathbf{B} \mathbf{u} + N u')) \end{pmatrix}, \quad (2.2)$$

$$A_2 V := \begin{pmatrix} \mathcal{L}^{-1} (N^T \mathbf{h}_E \tilde{\mathbf{G}}_E (\mathbf{h}_E^{-1} \mathbf{B} \mathbf{v}' + N v'')) \\ \mathbf{h}_{\mathcal{O}}^{-1} \mathbf{p}_{\mathcal{O}}^{-1} (-\mathbf{B}^T \tilde{\mathbf{G}}_E (\mathbf{h}_E^{-1} \mathbf{B} \mathbf{v} + N v')) \end{pmatrix}.$$

Let  $\langle u, v \rangle_{\Omega} = \int_{\Omega} u \cdot \bar{v} \, dx$  where  $u$  and  $v$  may be scalar or vector valued. Define the bilinear forms  $a$  and  $c$  by

$$\begin{aligned} c(z, v_{\mathcal{O}}; \hat{z}, \hat{v}_{\mathcal{O}}) &= m \langle z, \hat{z} \rangle_{\Omega} + \alpha \langle z', \hat{z}' \rangle_{\Omega} + \langle \mathbf{h}_{\mathcal{O}} \mathbf{p}_{\mathcal{O}} v_{\mathcal{O}}, \hat{v}_{\mathcal{O}} \rangle_{\Omega}, \\ a(z, v_{\mathcal{O}}; \hat{z}, \hat{v}_{\mathcal{O}}) &= K \langle z'', \hat{z}'' \rangle_{\Omega} + \langle \mathbf{h}_{\mathcal{O}} \mathbf{E}_{\mathcal{O}} v'_{\mathcal{O}}, \hat{v}'_{\mathcal{O}} \rangle_{\Omega} + \left\langle \mathbf{G}_E \mathbf{h}_E \phi_E, \hat{\phi}_E \right\rangle_{\Omega} \\ &= K \langle z'', \hat{z}'' \rangle_{\Omega} + \langle \mathbf{h}_{\mathcal{O}} \mathbf{E}_{\mathcal{O}} v'_{\mathcal{O}}, \hat{v}'_{\mathcal{O}} \rangle_{\Omega} \\ &\quad + \left\langle \mathbf{G}_E \mathbf{h}_E^{-1} (\mathbf{B} v_{\mathcal{O}} + N z'), (\mathbf{B} \hat{v}_{\mathcal{O}} + N \hat{z}') \right\rangle_{\Omega}. \end{aligned} \quad (2.3)$$

The “higher order” and natural energies of the beam are respectively given by

$$\mathcal{E}(t) = \begin{cases} \frac{1}{2} (a(z', v'_{\mathcal{O}}) + c(\dot{z}', \dot{v}'_{\mathcal{O}})) & \text{(h-N)} \end{cases} \quad (2.4a)$$

$$\begin{cases} \frac{1}{2} (a(z, v_0) + c(\dot{z}, \dot{v}_{\mathcal{O}})) & \text{(c,D), (m-m),} \end{cases} \quad (2.4b)$$

where  $a(\cdot), c(\cdot)$  are the quadratic forms that agree with  $a(\cdot, \cdot), c(\cdot, \cdot)$  on the diagonal. Define the energy inner products corresponding to each set of boundary conditions by

$$\langle Y, \hat{Y} \rangle_{\mathcal{H}} = \begin{cases} a(U'; \hat{U}') + c(V'; \hat{V}'). & \text{(h-N)} \end{cases} \quad (2.5a)$$

$$\begin{cases} a(U; \hat{U}) + c(V; \hat{V}) & \text{(c-D), (m-m).} \end{cases} \quad (2.5b)$$

Corresponding to each case, define the Hilbert spaces

$$\mathcal{H} = \begin{cases} H_*^3(\Omega) \times (H_{\perp}^2(\Omega))^{(m+1)} \times (H^2(\Omega) \cap H_0^1(\Omega)) \times (H_{\perp}^1(\Omega))^{(m+1)} & \text{(h-N)} \end{cases} \quad (2.6a)$$

$$\begin{cases} H_0^2(\Omega) \times (H_0^1(\Omega))^{(m+1)} \times H_0^1(\Omega) \times (L^2(\Omega))^{(m+1)} & \text{(c-D)} \end{cases} \quad (2.6b)$$

$$\begin{cases} H_{\#}^2(\Omega) \times (H_{\dagger}^1(\Omega))^{(m+1)} \times (H_0^1(\Omega)) \times (L^2(\Omega))^{(m+1)} & \text{(m-m)} \end{cases} \quad (2.6c)$$

where  $H_{\#}^2(\Omega)$  and  $H_{\dagger}^1(\Omega)$  are defined in (1.7) and

$$\begin{aligned} H_*^3(\Omega) &:= \{u \in H^3(\Omega) \cap H_0^1(\Omega) : u''(0) = u''(L) = 0\} \\ H_{\perp}^1(\Omega) &:= \{u \in H^1(\Omega) : \int_{\Omega} u \, dx = 0\} \\ H_{\perp}^2(\Omega) &:= \{u \in H^2(\Omega) \cap H_{\perp}^1(\Omega) : u'(0) = u'(L) = 0\}. \end{aligned}$$

Define  $\mathcal{D}(\mathcal{A})$  by

$$\mathcal{D}(\mathcal{A}) = \begin{cases} (H^4(\Omega) \cap H_*^3(\Omega)) \times (H^3(\Omega) \cap H_{\perp}^2(\Omega))^{(m+1)} \times H_*^3(\Omega) \times (H_{\perp}^2(\Omega))^{(m+1)} & \text{(h-N)} \\ (H^3(\Omega) \cap H_0^2(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega))^{(m+1)} \times H_0^2(\Omega) \times (H_0^1(\Omega))^{(m+1)} & \text{(c-D)} \\ H_{\#}^3(\Omega) \times (H_{\dagger}^2(\Omega))^{(m+1)} \times H_{\#}^2(\Omega) \times (H_{\dagger}^1(\Omega))^{(m+1)} & \text{(m-m)} \end{cases}$$

where

$$\begin{aligned} H_{\#}^3(\Omega) &:= \{u \in H_{\#}^2(\Omega) : u''(L) = 0\}, \\ H_{\dagger}^2(\Omega) &:= \{u \in H^2(\Omega) \cap H_{\dagger}^1(\Omega) : u'(L) = 0\}. \end{aligned}$$

LEMMA 2.1. *The operator  $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$  is densely defined.*

**Proof:** The density is obvious. However, in the case of hinged-Neumann boundary conditions (h-N), it is not obvious that the orthogonality constraint in the definition of  $\mathcal{H}$  is invariant with respect to  $\mathcal{A}$ , i.e., that  $Y \in \mathcal{D}(\mathcal{A})$  implies  $\mathcal{A}Y \in \mathcal{H}$ . To verify this, let  $Y = (u, \mathbf{u}, v, \mathbf{v})^T \in \mathcal{D}(\mathcal{A})$ . Then

$$(u, \mathbf{u}, v, \mathbf{v})^T \in (H^4(\Omega) \cap H_*^3(\Omega)) \times (H^3(\Omega) \cap H_{\perp}^2(\Omega))^{(m+1)} \times H_*^3(\Omega) \times (H_{\perp}^2(\Omega))^{(m+1)}.$$

From (2.1),  $\mathcal{A}Y = \begin{pmatrix} V \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -A_1U + A_2V \end{pmatrix}$ . Since  $v \in H_*^3(\Omega)$  and  $\mathbf{v} \in (H_{\perp}^2(\Omega))^{(m+1)}$ ,  $\begin{pmatrix} V \\ 0 \end{pmatrix} \in \mathcal{H}$ . Explicitly,  $-A_1U + A_2V$  is

$$\begin{pmatrix} \mathcal{L}^{-1} \left( -Ku'''' + N^T \mathbf{h}_E \left[ \mathbf{G}_E(\mathbf{h}_E^{-1} \mathbf{B} \mathbf{u}' + Nu'') + \tilde{\mathbf{G}}_E(\mathbf{h}_E^{-1} \mathbf{B} \mathbf{v}' + Nv'') \right] \right) \\ \mathbf{h}_O^{-1} \mathbf{p}_O^{-1} \left( \mathbf{h}_O \mathbf{E}_O \mathbf{u}'' - \mathbf{B}^T \left[ \mathbf{G}_E(\mathbf{h}_E^{-1} \mathbf{B} \mathbf{u} + Nu') - \tilde{\mathbf{G}}_E(\mathbf{h}_E^{-1} \mathbf{B} \mathbf{v} + Nv') \right] \right) \end{pmatrix}. \quad (2.8)$$

The first entry of (2.8) is in  $(H^2(\Omega) \cap H_0^1(\Omega))$  since  $\mathcal{L}^{-1}$  maps  $L^2(\Omega)$  to  $(H^2(\Omega) \cap H_0^1(\Omega))$ . Lastly, the second entry of (2.8) is in  $(H_{\perp}^1(\Omega))^{(m+1)}$  since the application of the (h-N) boundary conditions implies  $\int_{\Omega} u' \, dx = \int_{\Omega} \mathbf{u}'' \, dx = 0$ . Furthermore, since  $Y \in \mathcal{D}(\mathcal{A})$ ,  $\int_{\Omega} \mathbf{u} \, dx = \int_{\Omega} \mathbf{v} \, dx = 0$ , it follows that

$$\int_{\Omega} \mathbf{h}_O^{-1} \mathbf{p}_O^{-1} \mathbf{B}^T \mathbf{G}_E \mathbf{h}_E^{-1} \mathbf{B} \mathbf{u} \, dx = \int_{\Omega} \mathbf{h}_O^{-1} \mathbf{p}_O^{-1} \mathbf{B}^T \tilde{\mathbf{G}}_E \mathbf{h}_E^{-1} \mathbf{B} \mathbf{v} \, dx = 0. \quad \square$$

LEMMA 2.2. *The infinitesimal generator  $\mathcal{A}$  for each set of boundary conditions is dissipative, and moreover it satisfies*

$$\operatorname{Re} \langle \mathcal{A}Y, Y \rangle_{\mathcal{H}} = \begin{cases} - \langle \tilde{\mathbf{G}}_E \Theta', \mathbf{h}_E^{-1} \Theta' \rangle_{\Omega} \leq 0, & \text{(h-N)} \quad (2.9a) \\ - \langle \tilde{\mathbf{G}}_E \Theta, \mathbf{h}_E^{-1} \Theta \rangle_{\Omega} \leq 0, & \text{(c-D), (m-m)} \quad (2.9b) \end{cases}$$

for all  $Y = (u, \mathbf{u}, v, \mathbf{v})^T \in \mathcal{D}(\mathcal{A})$  where  $\Theta = (\mathbf{B}\mathbf{v} + \mathbf{h}_E N v')^T$ .

**Proof:** It is easy to show that  $\mathcal{A}$  is dissipative on  $\mathcal{H}$  for each set of boundary conditions. For example, consider the (h-N) boundary conditions:

$$\begin{aligned} \langle \mathcal{A}Y, Y \rangle_{\mathcal{H}} &= \{-K \langle u''', v''' \rangle_{\Omega} + K \langle v''', u''' \rangle_{\Omega}\} + \{-\langle \mathbf{h}_{\mathcal{O}} \mathbf{E}_{\mathcal{O}} \mathbf{u}'', \mathbf{v}'' \rangle_{\Omega} + \langle \mathbf{h}_{\mathcal{O}} \mathbf{E}_{\mathcal{O}} \mathbf{v}'', \mathbf{u}'' \rangle_{\Omega}\} \\ &\quad + \{-\langle \mathbf{G}_E (\mathbf{B}\mathbf{u}' + \mathbf{h}_E N u''), \mathbf{h}_E^{-1} (\mathbf{B}\mathbf{v}' + \mathbf{h}_E^{-1} N v'') \rangle_{\Omega} \\ &\quad + \langle \mathbf{G}_E (\mathbf{B}\mathbf{v}' + \mathbf{h}_E N v''), \mathbf{h}_E^{-1} (\mathbf{B}\mathbf{u}' + \mathbf{h}_E N u'') \rangle_{\Omega}\} \\ &\quad - \left\langle \tilde{\mathbf{G}}_E (\mathbf{B}\mathbf{v}' + \mathbf{h}_E N v''), \mathbf{h}_E^{-1} (\mathbf{B}\mathbf{v}' + \mathbf{h}_E N v'') \right\rangle_{\Omega} \\ &= -2i \operatorname{Im} (K \langle u''', v''' \rangle_{\Omega}) - 2i \operatorname{Im} (\langle \mathbf{h}_{\mathcal{O}} \mathbf{E}_{\mathcal{O}} \mathbf{u}'', \mathbf{v}'' \rangle_{\Omega}) \\ &\quad - 2i \operatorname{Im} \langle \mathbf{G}_E (\mathbf{B}\mathbf{u}' + \mathbf{h}_E N u''), \mathbf{h}_E^{-1} (\mathbf{B}\mathbf{v}' + \mathbf{h}_E^{-1} N v'') \rangle_{\Omega} - \left\langle \tilde{\mathbf{G}}_E \Theta', \mathbf{h}_E^{-1} \Theta' \right\rangle_{\Omega}. \end{aligned}$$

Therefore (2.9) follows.  $\square$

LEMMA 2.3.  $I - \mathcal{A} : \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{H}$  is surjective.

**Proof:** We prove the lemma for only (h-N) boundary conditions since the proofs for other boundary conditions are similar. Let  $C$  denote a generic constant in the following calculations, and define  $|u|_s = \|u\|_{H^s(\Omega)}$ ,  $|\mathbf{u}|_s = \|\mathbf{u}\|_{(H^s(\Omega))^{(m+1)}}$ . Let  $Y_1 = (u_1, \mathbf{u}_1, v_1, \mathbf{v}_1)^T$ . For given  $Y_2 = (u_2, \mathbf{u}_2, v_2, \mathbf{v}_2)^T \in \mathcal{H}$  we want to prove the solvability of the system  $(I - \mathcal{A})Y_1 = Y_2$  in  $\mathcal{D}(\mathcal{A})$ :

$$\begin{aligned} K u_1'''' - N^T \mathbf{h}_E \left( \mathbf{G}_E (\mathbf{h}_E^{-1} \mathbf{B} \mathbf{u}_1' + N u_1'') + \tilde{\mathbf{G}}_E (\mathbf{h}_E^{-1} \mathbf{B} \mathbf{v}_1' + N v_1'') \right) &= \mathcal{L} v_2 - \mathcal{L} v_1 \\ -\mathbf{h}_{\mathcal{O}} \mathbf{E}_{\mathcal{O}} \mathbf{u}_1'' + \mathbf{B}^T \left( \mathbf{G}_E (\mathbf{h}_E^{-1} \mathbf{B} \mathbf{u}_1 + N u_1') + \tilde{\mathbf{G}}_E (\mathbf{h}_E^{-1} \mathbf{B} \mathbf{v}_1 + N v_1') \right) &= \mathbf{p}_{\mathcal{O}} \mathbf{h}_{\mathcal{O}} (\mathbf{v}_2 - \mathbf{v}_1) \\ u_1 - v_1 &= u_2 \\ \mathbf{u}_1 - \mathbf{v}_1 &= \mathbf{u}_2. \end{aligned} \quad (2.10)$$

Differentiating the second equation in (2.10) yields

$$\begin{aligned} K u_1'''' - N^T \mathbf{h}_E \left( \mathbf{G}_E (\mathbf{h}_E^{-1} \mathbf{B} \mathbf{u}_1' + N u_1'') + \tilde{\mathbf{G}}_E (\mathbf{h}_E^{-1} \mathbf{B} \mathbf{v}_1' + N v_1'') \right) &= \mathcal{L} v_2 - \mathcal{L} v_1 \\ -\mathbf{h}_{\mathcal{O}} \mathbf{E}_{\mathcal{O}} \mathbf{u}_1''' + \mathbf{B}^T \left( \mathbf{G}_E (\mathbf{h}_E^{-1} \mathbf{B} \mathbf{u}_1' + N u_1'') + \tilde{\mathbf{G}}_E (\mathbf{h}_E^{-1} \mathbf{B} \mathbf{v}_1' + N v_1'') \right) &= \mathbf{p}_{\mathcal{O}} \mathbf{h}_{\mathcal{O}} (\mathbf{v}_2' - \mathbf{v}_1') \\ u_1 - v_1 &= u_2 \\ \mathbf{u}_1 - \mathbf{v}_1 &= \mathbf{u}_2. \end{aligned} \quad (2.11)$$

We eliminate the functions  $v_1, \mathbf{v}_1$  from the last two equations in (2.11). Then, we multiply the first equation  $u_1''''$  and the second by  $\mathbf{u}_1'''$ , and integrate by parts on  $\Omega$ , using boundary conditions for  $\mathcal{D}(\mathcal{A})$ , and then we eventually use Holder's inequality to obtain the following estimate:

$$\begin{aligned} |u_1|_4 &\leq C (|u_1|_2 + |\mathbf{u}_1|_1 + |u_2|_2 + |v_2|_2 + |\mathbf{u}_2|_1) \\ |\mathbf{u}_1|_3 &\leq C (|u_1|_2 + |\mathbf{u}_1|_1 + |u_2|_2 + |\mathbf{u}_2|_2 + |\mathbf{v}_2|_1) \\ |v_1|_3 &\leq C (|u_1|_3 + |u_2|_3) \\ |\mathbf{v}_1|_2 &\leq C (|\mathbf{u}_1|_2 + |\mathbf{u}_2|_2). \end{aligned} \quad (2.12)$$

The next step is to absorb the lower order terms in (2.12) to get

$$|u_1|_4 + |\mathbf{u}_1|_3 + |v_1|_3 + |\mathbf{v}_1|_2 \leq C (|u_2|_3 + |\mathbf{u}_2|_2 + |v_2|_2 + |\mathbf{v}_2|_1). \quad (2.13)$$

We apply a standard compactness-uniqueness argument: now suppose contrarily that the inequality (2.13) does not hold. Then there exists a sequence  $Y_{2n} := \{(u_{2n}, \mathbf{u}_{2n}, v_{2n}, \mathbf{v}_{2n})^T\}_{n=1}^\infty$  such that

$$\|Y_{2n}\|_{\mathcal{H}} \rightarrow 0, \quad \text{and} \quad |u_{1n}|_4 + |\mathbf{u}_{1n}|_3 + |v_{1n}|_3 + |\mathbf{v}_{1n}|_2 = 1. \quad (2.14)$$

From (2.14) we can extract a subsequence, still denoted  $Y_{1n} := \{(u_{1n}, \mathbf{u}_{1n}, v_{1n}, \mathbf{v}_{1n})^T\}_{n=1}^\infty$  such that  $Y_{1n}$  converges to  $Y_1 := (u_1, \mathbf{u}_1, v_1, \mathbf{v}_1)$  weakly in  $H^4(\Omega) \times (H^3(\Omega))^{(m+1)} \times H^3(\Omega) \times (H^2(\Omega))^{(m+1)} := \mathcal{W}$ . If we consider the solution of (2.10) with  $Y_{1n} = Y_{1n}(Y_{2n})$ , then it follows from (2.12) that

$$\begin{aligned} |u_{1n} - u_{1m}|_4 &\leq C(|u_{1n} - u_{1m}|_2 + |\mathbf{u}_{1n} - \mathbf{u}_{1m}|_1 + |u_{2n} - u_{2m}|_2 \\ &\quad + |v_{2n} - v_{2m}|_2 + |\mathbf{u}_{2n} - \mathbf{u}_{2m}|_1) \\ |\mathbf{u}_{1n} - \mathbf{u}_{1m}|_3 &\leq C(|u_{1n} - u_{1m}|_2 + |\mathbf{u}_{1n} - \mathbf{u}_{1m}|_1 + |u_{2n} - u_{2m}|_2 \\ &\quad + |\mathbf{u}_{2n} - \mathbf{u}_{2m}|_2 + |\mathbf{v}_{2n} - \mathbf{v}_{2m}|_1) \\ |v_{1n} - v_{1m}|_3 &\leq C(|u_{1n} - u_{1m}|_3 + |u_{2n} - u_{2m}|_3) \\ |\mathbf{v}_{1n} - \mathbf{v}_{1m}|_2 &\leq C(|\mathbf{u}_{1n} - \mathbf{u}_{1m}|_2 + |\mathbf{u}_{2n} - \mathbf{u}_{2m}|_2). \end{aligned}$$

Thus, by the Sobolev's compact embedding theorem we get

$$|u_{1n} - u_{1m}|_4, |\mathbf{u}_{1n} - \mathbf{u}_{1m}|_3, |v_{1n} - v_{1m}|_3, |\mathbf{v}_{1n} - \mathbf{v}_{1m}|_2 \rightarrow 0,$$

as  $n, m \rightarrow \infty$ . This implies that  $Y_{1n}$  actually converges to  $Y_1$  strongly in  $\mathcal{W}$ . On the other hand, the system (2.10) with  $Y_2 = (0, \mathbf{0}, 0, \mathbf{0})^T$ , see (2.14), has only a trivial solution since the system (2.1) is dissipative by (2.9). This contradicts with (2.14) and therefore (2.13) holds. Hence  $Y_1 \in \mathcal{D}(\mathcal{A})$  and the claim of the theorem is proved.

**THEOREM 2.1.**  $\mathcal{A} : \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{H}$  is the infinitesimal generator of a  $C_0$ -semigroup of contractions. Moreover, the spectrum of  $\mathcal{A}$  only consists of isolated non-zero eigenvalues  $\{\gamma_n\}_{n=1}^\infty$ , and  $|\gamma_n^\pm| \rightarrow \infty$  as  $n \rightarrow \infty$ .

**Proof:** The proof of the first part follows from the L umer-Phillips theorem [15] using Lemma 2.1, 2.2 and 2.3. Since  $(\mathcal{I} - \mathcal{A})^{-1}$  is compact, the spectrum of  $\mathcal{A}$  only consists of eigenvalues. A simple proof that  $0 \in \rho(\mathcal{A})$  for the (h-N) case ( $m = 1$ ) is given in [4]. The same proof applies for any positive integer  $m$  and also the boundary conditions (c-D) and (m-m). Hence the claim of the theorem follows.  $\square$

**COROLLARY 2.1.** The operator  $\mathcal{A}^* : \mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A}^*) \rightarrow \mathcal{H}$  is the generator of a  $C_0$ -contraction semigroup. Moreover,

$$\left[ \mathcal{A}(\tilde{\mathbf{G}}_E) \right]^* = -\mathcal{A}(-\tilde{\mathbf{G}}_E), \quad \text{on } \mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A}^*)$$

where  $\mathcal{A}(\tilde{\mathbf{G}}_E)$  denotes the dependence of  $\mathcal{A}$  on the parameter  $\tilde{\mathbf{G}}_E$ .

**Proof:** A straightforward (but lengthy) calculation shows that  $\left[ \mathcal{A}(\tilde{\mathbf{G}}_E) \right]^* = -\mathcal{A}(-\tilde{\mathbf{G}}_E)$  on  $\mathcal{D}(\mathcal{A})$  for each of the sets of boundary conditions considered. Moreover  $-\mathcal{A}(-\tilde{\mathbf{G}}_E)$  is dissipative by (2.9). Thus the proof of Lemma 2.3 remains valid with  $-\mathcal{A}(-\tilde{\mathbf{G}}_E)$  in place of  $\mathcal{A}$ . Since  $\mathcal{I} + \mathcal{A}(-\tilde{\mathbf{G}}_E) : \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{H}$  is bijective,  $\mathcal{D}(\mathcal{A}^*)$  can be no larger than  $\mathcal{D}(\mathcal{A})$ . Thus,  $\mathcal{D}(\mathcal{A}^*) = \mathcal{D}(\mathcal{A})$ . It follows from the corollary of L umer-Phillips theorem ([15], Chap I) that  $\mathcal{A}^*$  generates a contraction semigroup.  $\square$



Let  $\mathcal{H}_{-1}$  be the dual space of  $\mathcal{H}_1 := \mathcal{D}(\mathcal{A})$  pivoted with respect to  $\mathcal{H}$ . Then we have the following dense and compact embeddings

$$\mathcal{H}_1 \subset \mathcal{H} \subset \mathcal{H}_{-1}.$$

By Proposition 2.10.3 in [19], the operator  $\mathcal{A} : \mathcal{H}_1 \rightarrow \mathcal{H}$  has a unique extension  $\tilde{\mathcal{A}} : \mathcal{H} \rightarrow \mathcal{H}_{-1}$  defined by

$$\langle \tilde{\mathcal{A}}Y, Z \rangle := \langle Y, \mathcal{A}^*Z \rangle_{\mathcal{H}}, \quad \forall Z \in \mathcal{H}_1, Y \in \mathcal{H}. \quad (2.15)$$

By Proposition 2.10.4 in [19],  $\tilde{\mathcal{A}}$  is the generator of a  $C_0$ -semigroup  $\{e^{\tilde{\mathcal{A}}t}\}_{t \geq 0}$  on  $\mathcal{H}_{-1}$ , which is similar to  $\{e^{\mathcal{A}t}\}_{t \geq 0}$ . Thus we have the following.

**COROLLARY 2.2.** *The semigroup  $\{e^{\mathcal{A}t}\}_{t \geq 0}$  with the generator  $\mathcal{A} : \mathcal{H}_1 \rightarrow \mathcal{H}$  has a unique extension to a contraction semigroup  $\{e^{\tilde{\mathcal{A}}t}\}_{t \geq 0}$  on  $\mathcal{H}_{-1}$  with the generator  $\tilde{\mathcal{A}} : \mathcal{H} \rightarrow \mathcal{H}_{-1}$ .*

**3. Characterization of the space  $\mathcal{H}_{-1}$  in undamped case.** In particular, we are interested in a characterization of the space  $\mathcal{H}_{-1}$  for the (m-m) boundary conditions. Define spaces  $\mathcal{X}_2, \mathcal{X}_1, \mathcal{X}$  by

$$\mathcal{X}_2 = \begin{cases} (H^4(\Omega) \cap H_*^3(\Omega)) \times (H^3(\Omega) \cap H_{\perp}^2(\Omega))^{(m+1)} & \text{(h-N)} \\ (H^3(\Omega) \cap H_0^2(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega))^{(m+1)} & \text{(c-D)} \\ H_{\#}^3(\Omega) \times (H_{\dagger}^2(\Omega))^{(m+1)} & \text{(m-m)} \end{cases}$$

$$\mathcal{X}_1 = \begin{cases} H_*^3(\Omega) \times (H_{\perp}^2(\Omega))^{(m+1)} & \text{(h-N)} \\ H_0^2(\Omega) \times (H_0^1(\Omega))^{(m+1)} & \text{(c-D)} \\ H_{\#}^2(\Omega) \times (H_{\dagger}^1(\Omega))^{(m+1)} & \text{(m-m)} \end{cases}$$

$$\mathcal{X} = \begin{cases} (H^2(\Omega) \cap H_0^1(\Omega)) \times (H_{\perp}^1(\Omega))^{(m+1)} & \text{(h-N)} \\ H_0^1(\Omega) \times (L^2(\Omega))^{(m+1)} & \text{(c-D), (m-m)}. \end{cases}$$

Also define the inner products

$$\langle U, V \rangle_{\mathcal{X}_1} = \begin{cases} a(U'; V') & \text{(h-N)} \\ a(U; V) & \text{(c-D), (m-m)}, \end{cases}$$

where  $U = (u, \mathbf{u})^T, V = (v, \mathbf{v})^T$  and the bilinear form  $a$  is defined in (2.3);

$$\langle U, V \rangle_{\mathcal{X}} = \begin{cases} c(U'; V') = m \langle u', v' \rangle_{\Omega} + \alpha \langle u'', v'' \rangle_{\Omega} + \langle \mathbf{h} \circ \mathbf{p} \circ \mathbf{u}', \mathbf{v}' \rangle_{\Omega} \\ \quad = -\langle \mathcal{L}u, v'' \rangle_{\Omega} + \langle \mathbf{h} \circ \mathbf{p} \circ \mathbf{u}', \mathbf{v}' \rangle_{\Omega}, & \text{(h-N)} \\ c(U; V) = m \langle u, v \rangle_{\Omega} + \alpha \langle u', v' \rangle_{\Omega} + \langle \mathbf{h} \circ \mathbf{p} \circ \mathbf{u}, \mathbf{v} \rangle_{\Omega} \\ \quad = -\langle \mathcal{L}u, v \rangle_{\Omega} + \langle \mathbf{h} \circ \mathbf{p} \circ \mathbf{u}, \mathbf{v} \rangle_{\Omega}, & \text{(c-D), (m-m)}. \end{cases} \quad (3.3)$$

Then

$$\mathcal{D}(\mathcal{A}) = \mathcal{X}_2 \times \mathcal{X}_1, \quad \mathcal{H} = \mathcal{X}_1 \times \mathcal{X}$$

and the inner product for  $\mathcal{H}$  can be written

$$\langle Y, \hat{Y} \rangle_{\mathcal{H}} = \langle (U, V)^T, (\hat{U}, \hat{V})^T \rangle_{\mathcal{H}} = \langle U, \hat{U} \rangle_{\mathcal{X}_1} + \langle V, \hat{V} \rangle_{\mathcal{X}}.$$

Let  $A_1$  be the operator on  $\mathcal{X}_1$  defined by (2.2). For each of the sets of boundary conditions (h-N), (m-m) or (c-D), a simple calculation establishes the following identity:

$$\langle A_1 U, V \rangle_{\mathcal{X}} = \langle U, V \rangle_{\mathcal{X}_1} \quad \forall U, V \in \mathcal{X}_2. \quad (3.4)$$

For instance, in the (h-N) case,

$$\begin{aligned} \langle A_1 U, V \rangle_{\mathcal{X}} &= \left\langle \begin{pmatrix} \mathcal{L}^{-1} (K u'''' - N^T \mathbf{h}_E \mathbf{G}_E \phi'_E) \\ \mathbf{h}_O^{-1} \mathbf{p}_O^{-1} (-\mathbf{h}_O \mathbf{E}_O \mathbf{u}'' + \mathbf{B}^T \mathbf{G}_E \phi_E) \end{pmatrix}, V \right\rangle_{\mathcal{X}} \\ &= \langle -K u'''' + N^T \mathbf{h}_E \mathbf{G}_E \phi'_E, v'' \rangle_{\Omega} + \langle -\mathbf{h}_O \mathbf{E}_O \mathbf{u}'' + \mathbf{B}^T \mathbf{G}_E \phi'_E, \mathbf{v}' \rangle_{\Omega} \\ &= K \langle u''', v''' \rangle_{\Omega} + \langle \mathbf{h}_O \mathbf{E}_O \mathbf{u}'', \mathbf{v}'' \rangle_{\Omega} + \langle \mathbf{G}_E \phi'_E, \mathbf{h}_E N v'' + \mathbf{B} \mathbf{v}' \rangle_{\Omega} \\ &= K \langle u''', v''' \rangle_{\Omega} + \langle \mathbf{h}_O \mathbf{E}_O \mathbf{u}'', \mathbf{v}'' \rangle_{\Omega} + \langle \mathbf{h}_E \mathbf{G}_E \phi'_E, \psi'_E \rangle_{\Omega} = \langle U, V \rangle_{\mathcal{X}_1}. \end{aligned}$$

Let  $\mathcal{X}_{-1}$  denote the dual of  $\mathcal{X}_1$  with respect to  $\mathcal{X}$ . By the Lax-Milgram theorem,  $A_1$  extends to an isomorphism between  $\mathcal{X}_1$  and  $\mathcal{X}_{-1}$ . Therefore, the inner product on  $\mathcal{X}$  extends continuously to the duality pairing  $\langle \cdot, \cdot \rangle_{\mathcal{X}_{-1}, \mathcal{X}_1}$  which satisfies (for  $U, V \in \mathcal{X}_1$ )

$$\langle A_1 U, V \rangle_{\mathcal{X}_{-1}, \mathcal{X}_1} = a(U'; V') = K \langle u''', v''' \rangle_{\Omega} + \langle \mathbf{h}_O \mathbf{E}_O \mathbf{u}'', \mathbf{v}'' \rangle_{\Omega} + \langle \mathbf{G}_E \mathbf{h}_E \phi'_E, \psi'_E \rangle_{\Omega}$$

for the (h-N) boundary conditions and

$$\langle A_1 U, V \rangle_{\mathcal{X}_{-1}, \mathcal{X}_1} = a(U; V) = K \langle u'', v'' \rangle_{\Omega} + \langle \mathbf{h}_O \mathbf{E}_O \mathbf{u}', \mathbf{v}' \rangle_{\Omega} + \langle \mathbf{G}_E \mathbf{h}_E \phi_E, \psi_E \rangle_{\Omega}$$

for the (c-D) and (m-m) boundary conditions. Furthermore, we have dense compact embeddings  $\mathcal{X}_1 \hookrightarrow \mathcal{X} \hookrightarrow \mathcal{X}_{-1}$ .

From (3.4),  $A_1$  is a positive and self-adjoint operator. Therefore there exists a sequence of orthogonal eigenvectors  $\{E_{k,l}\} \in \mathcal{X}_1, k \geq 1, 1 \leq l \leq m_k$  corresponding to the eigenvalues  $\lambda_k$  and

$$\begin{aligned} A_1 E_{k,l} &= \lambda_k E_{k,l}, \quad 1 \leq l \leq m_k \\ \lambda_k &> 0, \quad \lambda_k \rightarrow \infty, \quad 1 \leq l \leq m_k \quad \text{as } k \rightarrow \infty, \quad E_{k,l} \perp E_{m,n} \text{ if } k \neq m. \end{aligned} \quad (3.5)$$

By (3.4), we have

$$\langle A_1 E_{k,l}, E_{k,l} \rangle_{\mathcal{X}} = \langle \lambda_k E_{k,l}, E_{k,l} \rangle_{\mathcal{X}} = \lambda_k \|E_{k,l}\|_{\mathcal{X}}^2 = \|E_{k,l}\|_{\mathcal{X}_1}^2.$$

Every  $U \in \mathcal{X}_1$  has a unique orthogonal expansion  $\sum_{k \geq 1, 1 \leq l \leq m_k} c_{k,l} E_{k,l}$  and it follows from (3.4) that we have

$$\|U\|_{\mathcal{X}_1}^2 = \sum_{k \geq 1, 1 \leq l \leq m_k} \|c_{k,l} E_{k,l}\|_{\mathcal{X}_1}^2 = \sum_{k \geq 1, 1 \leq l \leq m_k} \lambda_k c_{k,l}^2 \|E_{k,l}\|_{\mathcal{X}}^2. \quad (3.6)$$

The inner product on  $\mathcal{X}_{-1}$  is defined by

$$\langle U, V \rangle_{\mathcal{X}_{-1}} = \langle A_1^{-1} U, A_1^{-1} V \rangle_{\mathcal{X}_1}. \quad (3.7)$$

Note that the eigenfunctions  $\{E_{k,l}\}_{k \geq 1, 1 \leq l \leq m_k}$  preserves their orthogonality in  $\mathcal{X}$  and  $\mathcal{X}_{-1}$ . Therefore, every  $U \in \mathcal{X}$  (or  $\mathcal{X}_{-1}$ ) has a unique orthogonal expansion of the form  $\sum_{k \geq 1, 1 \leq l \leq m_k} c_{k,l} E_{k,l}$  converging in  $\mathcal{X}$  (or  $\mathcal{X}_{-1}$ ), and we have

$$\|U\|_{\mathcal{X}}^2 = \sum_{k \geq 1, 1 \leq l \leq m_k} c_{k,l}^2 \|E_{k,l}\|_{\mathcal{X}}^2,$$

and respectively

$$\begin{aligned} \|U\|_{\mathcal{X}_{-1}}^2 &= \sum_{k \geq 1, 1 \leq l \leq m_k} c_{k,l}^2 \|E_{k,l}\|_{\mathcal{X}_{-1}}^2 = \sum_{k \geq 1, 1 \leq l \leq m_k} c_{k,l}^2 \|A_1^{-1} E_{k,l}\|_{\mathcal{X}_1}^2 \\ &= \sum_{k \geq 1, 1 \leq l \leq m_k} \lambda_k^{-2} c_{k,l}^2 \|E_{k,l}\|_{\mathcal{X}_1}^2 = \sum_{k \geq 1, 1 \leq l \leq m_k} \lambda_k^{-1} c_{k,l}^2 \|E_{k,l}\|_{\mathcal{X}}^2. \end{aligned} \quad (3.8)$$

Eq. (3.8) provides one characterization of  $\mathcal{X}_{-1}$ . However, we would like a function space characterization, particularly in the case of (m-m) boundary conditions.

We will need to refer Lemmata 3.1 and 3.2 below, which are proved in [14], and are adaptations of similar results in [9].

**LEMMA 3.1.** *Let  $H = \text{span} \left\{ \sinh \frac{x-L}{\sqrt{\alpha/m}} \right\} \subset L^2(\Omega)$ . Let  $\mathcal{L}$  be the operator  $mI - \alpha D_x^2$  on the domain  $H^2(\Omega) \cap H_0^1(\Omega)$ . Then the restriction of  $\mathcal{L}$  to  $H_{\#}^2(\Omega)$  is an isomorphism from  $H_{\#}^2(\Omega)$  to  $H^{\perp}$  in  $L^2(\Omega)$ .*

**LEMMA 3.2.**  $H^{\perp} = (L^2(\Omega)/H)'$ , where the duality is with respect to the  $L^2(\Omega)$  inner product.

Now consider specifically the (m-m) boundary conditions. For  $V = (v, \mathbf{v}) \in \mathcal{X}_1 = H_{\#}^2(\Omega) \times (H_{\dagger}^1(\Omega))^{(m+1)}$ ,  $U = (u, \mathbf{u}) \in \mathcal{X} = H_0^1(\Omega) \times (L^2(\Omega))^{(m+1)}$ , an integration by parts of (3.3) results in

$$c(U, V) = -\langle u, \mathcal{L}v \rangle_{\Omega} + \langle \mathbf{h} \circ \mathbf{p} \circ \mathbf{u}, \mathbf{v} \rangle_{\Omega}.$$

The second term remains bounded for all  $\mathbf{u} \in (H_{\dagger}^1(\Omega))^{(m+1)'} (with duality relative to  $L^2(\Omega)$ ). In the first term, however, by Lemma 3.1, the range of  $\mathcal{L}$  is  $H^{\perp}$  in  $L^2(\Omega)$ . Hence for the first term to remain bounded, by Lemma 3.2,  $u \in L^2(\Omega)/H$ . Therefore, in the case of (m-m) boundary conditions,$

$$\mathcal{X}_{-1} = L^2(\Omega)/H \times (H_{\dagger}^1(\Omega))^{(m+1)'} \quad (3.9)$$

It is easiest to characterize  $\mathcal{H}_{-1}$  in the undamped case. (Later we will show that the same characterization holds in the damped case.) Write the operator  $\mathcal{A}$  as follows:

$$\mathcal{A} = \mathcal{A}_0 + \mathcal{B} = \begin{pmatrix} 0 & I \\ -A_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & A_2 \end{pmatrix}$$

Then  $\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A}_0)$  and hence  $\mathcal{A}_0 : \mathcal{H} = \mathcal{X}_1 \times \mathcal{X} \rightarrow \mathcal{H}_{-1}$  is an isomorphism by Theorem 2.1 and Corollary 2.2. It follows that an inner product on  $\mathcal{H}_{-1}$  can be defined by  $\langle Y, Z \rangle_{\mathcal{H}_{-1}} = \langle \mathcal{A}_0^{-1} Y, \mathcal{A}_0^{-1} Z \rangle_{\mathcal{H}}$ . Hence, in the undamped case,

$$\begin{aligned} \langle Y, Z \rangle_{\mathcal{H}_{-1}} &= \langle \mathcal{A}_0^{-1} Y, \mathcal{A}_0^{-1} Z \rangle_{\mathcal{H}} \\ &= c(Y_1, Z_1) + a(-A_1^{-1} Y_2, -A_1^{-1} Z_2) \\ &= \langle Y_1, Z_1 \rangle_{\mathcal{X}} + \langle A_1^{-1} Y_2, A_1^{-1} Z_2 \rangle_{\mathcal{X}_1} \\ &= \langle Y_1, Z_1 \rangle_{\mathcal{X}} + \langle Y_2, Z_2 \rangle_{\mathcal{X}_{-1}} \end{aligned}$$

where we used (2.5) and (3.7). By (3.9), we have in the undamped case with (m-m) boundary conditions,

$$\mathcal{H}_{-1} = \mathcal{X} \times \mathcal{X}_{-1} = H_0^1(\Omega) \times (L^2(\Omega))^{(m+1)} \times (L^2(\Omega)/H) \times (H_1^1(\Omega)')^{(m+1)}. \quad (3.10)$$

**4. Observability results and the Proof of Theorem 1.2.** We prove our main observability results in this section. We begin with some preliminary results for the decoupled system.

**4.1. Observability results for decoupled system.** Consider (1.9) without the coupling terms, i.e., with  $\mathbf{G}_E = \tilde{\mathbf{G}}_E = 0$ . What remains is a Rayleigh beam equation and  $(m+1)$  wave equations:

$$\begin{cases} m\ddot{z} - \alpha\ddot{z}'' + Kz'''' = 0 & \text{on } \Omega \times \mathbb{R}^+ \\ \ddot{v}_{\mathcal{O}} - \mathbf{p}_{\mathcal{O}}^{-1}\mathbf{E}_{\mathcal{O}}v_{\mathcal{O}}'' = 0 & \text{on } \Omega \times \mathbb{R}^+, \end{cases} \quad (4.1)$$

with the boundary conditions (1.10) and the initial conditions (1.11). Let

$$U =: (u, \mathbf{u}) = (z, v_{\mathcal{O}})^T, \quad V := (v, \mathbf{v})^T = (\dot{z}, \dot{v}_{\mathcal{O}})^T, \quad \text{and } Y := (U, V)^T.$$

Then the semigroup corresponding to (4.1) is given by

$$\begin{aligned} \frac{dY}{dt} &= \mathcal{A}_d Y := \begin{pmatrix} 0 & I \\ -A_d & 0 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}, \\ Y(0) &= (U(0), V(0))^T = (z^0, v_{\mathcal{O}}^0, z^1, v_{\mathcal{O}}^1)^T \end{aligned}$$

where  $A_d U := \begin{pmatrix} K\mathcal{L}^{-1}u'''' \\ -\mathbf{p}_{\mathcal{O}}^{-1}\mathbf{E}_{\mathcal{O}}u'' \end{pmatrix}$ . Define the quadratic forms  $a_d$  and  $c_d$  by

$$\begin{aligned} c_d(z, v_{\mathcal{O}}) &= m\langle z, z \rangle_{\Omega} + \alpha\langle z', z' \rangle_{\Omega} + \langle \mathbf{h}_{\mathcal{O}}\mathbf{p}_{\mathcal{O}}v_{\mathcal{O}}, v_{\mathcal{O}} \rangle_{\Omega} \\ a_d(z, v_{\mathcal{O}}) &= K\langle z'', z'' \rangle_{\Omega} + \langle \mathbf{h}_{\mathcal{O}}\mathbf{E}_{\mathcal{O}}v'_{\mathcal{O}}, v'_{\mathcal{O}} \rangle_{\Omega}. \end{aligned}$$

The natural and “higher order” energies of the decoupled system are given by

$$\mathcal{E}_d(t) = \begin{cases} \frac{1}{2}(a_d(z', v'_{\mathcal{O}}) + c_d(\dot{z}', \dot{v}'_{\mathcal{O}})) & \text{(h-N)} \\ \frac{1}{2}(a_d(z, v_{\mathcal{O}}) + c_d(\dot{z}, \dot{v}_{\mathcal{O}})). & \text{(c-D), (m-m).} \end{cases}$$

The energy inner products corresponding to each set of boundary conditions are defined by

$$\langle Y, \hat{Y} \rangle_{\mathcal{H}} = \begin{cases} a_d(U'; \hat{U}') + c_d(V'; \hat{V}') & \text{(h-N)} \\ a_d(U; \hat{U}) + c_d(V; \hat{V}) & \text{(c-D), (m-m).} \end{cases}$$

In the above  $\mathcal{A}_d$  is densely defined by  $\mathcal{A}_d : \mathcal{D}(\mathcal{A}_d) \subset \mathcal{H} \rightarrow \mathcal{H}$  and note that  $\mathcal{D}(\mathcal{A}_d) = \mathcal{D}(\mathcal{A})$ .

**REMARK 4.1.** (i) It is easy to verify that  $\mathcal{E}(t) \asymp \mathcal{E}_d(t)$ ,  $\forall t > 0$ . Indeed, for the hinged-Neumann (h-N) boundary conditions

$$\begin{aligned} |\langle \mathbf{G}_E \mathbf{h}_E \phi'_E, \phi'_E \rangle_{\Omega}| &= |\langle \mathbf{G}_E \mathbf{h}_E^{-1} (\mathbf{B}v'_{\mathcal{O}} + \mathbf{h}_E N z''), (\mathbf{B}v'_{\mathcal{O}} + \mathbf{h}_E N z'') \rangle_{\Omega}| \\ &\leq C \left( \|v''_{\mathcal{O}}\|_{(L^2(\Omega))^{(m+1)}}^2 + \|z'''\|_{L^2(\Omega)}^2 \right) \leq C \mathcal{E}_d, \end{aligned}$$

and for the clamped-Dirichlet (c-D) and mixed-mixed (m-m) boundary conditions

$$\begin{aligned} |\langle \mathbf{G}_E \mathbf{h}_E \phi_E, \phi_E \rangle_\Omega| &= |\langle \mathbf{G}_E \mathbf{h}_E^{-1} (\mathbf{B}v_\mathcal{O} + \mathbf{h}_E N z'), (\mathbf{B}v_\mathcal{O} + \mathbf{h}_E N z') \rangle_\Omega| \\ &\leq C \left( \|v'_\mathcal{O}\|_{(L^2(\Omega))^{(m+1)}}^2 + \|z''\|_{L^2(\Omega)}^2 \right) \leq C \mathcal{E}_d \end{aligned}$$

where  $C$  denotes a generic constant. Therefore,

$$\mathcal{E}_d \leq \mathcal{E} \leq C \mathcal{E}_d. \quad (4.4)$$

(ii) In the case of (m-m) boundary conditions, we define the solutions of (4.1), (1.10) and (1.11) on the extended space  $\mathcal{H}_{-1}$  (defined by (3.10)) in exactly the same way as we did for the undamped coupled system, i.e., by applying Corollary 2.2, Lemma 3.1, and Lemma 3.2 to the decoupled system. Therefore we define the energy of the weak solutions by

$$\begin{aligned} \mathcal{E}_{-1}(t) &= \frac{1}{2} \|(z, \dot{z}, v_\mathcal{O}, \dot{v}_\mathcal{O})\|_{\mathcal{H}_{-1}}^2 \\ &\approx \frac{1}{2} \left( \|z\|_{H_0^1(\Omega)}^2 + \|\dot{z}\|_{(L^2(\Omega))^{(m+1)}}^2 + \|v_\mathcal{O}\|_{L^2(\Omega)/\mathbf{H}}^2 + \|\dot{v}_\mathcal{O}\|_{((H_+^1(\Omega))')^{(m+1)}}^2 \right). \end{aligned} \quad (4.5)$$

The following results for the interior regularity, hidden regularity, and observability of the decoupled system (4.1) follow from the standard semigroup theory, standard results for the wave equation, e.g. see [6], [10], and observability results obtained in [14].

THEOREM 4.1.

(a) Consider

$$\begin{cases} m\ddot{z} - \alpha\ddot{z}'' + Kz'''' + f(x, t) = 0 & \text{in } \Omega \times \mathbb{R}^+ \\ \ddot{v}_\mathcal{O} - \mathbf{p}_\mathcal{O}^{-1} \mathbf{E}_\mathcal{O} v_\mathcal{O}'' + f_\mathcal{O}(x, t) = 0 & \text{in } \Omega \times \mathbb{R}^+ \end{cases} \quad (4.6)$$

with the boundary conditions (1.10) and the initial conditions

$$z(x, 0) = \dot{z}(x, 0) = 0, \quad v_\mathcal{O}(x, 0) = \dot{v}_\mathcal{O}(x, 0) = 0 \text{ on } \Omega.$$

Assume

$$\begin{cases} f \in L^1(0, T; L^2(\Omega)), \quad f_\mathcal{O} \in L^1(0, T; (H^1(\Omega))^{(m+1)}) & (h-N) \\ f \in L^1(0, T; H^{-1}(\Omega)), \quad f_\mathcal{O} \in L^1(0, T; (L^2(\Omega))^{(m+1)}) & (c-D) \\ f \in L^1(0, T; L^2(\Omega)/\mathbf{H}), \quad f_\mathcal{O} \in L^1(0, T; ((H_+^1(\Omega))')^{(m+1)}) & (m-m). \end{cases}$$

Then  $(z, \dot{z}, v_\mathcal{O}, \dot{v}_\mathcal{O}) \in C([0, T]; \mathcal{H})$  and the solution of (4.6) satisfy for every  $T > 0$  the direct inequality

$$\begin{aligned} \int_0^T (|z'''(L, t)|^2 + |v_\mathcal{O}''(L, t)|^2) \, dt &\leq C \|(f, f'_\mathcal{O})\|_{L^1(0, T; L^2(\Omega) \times (L^2(\Omega))^{(m+1)})}^2 \\ \int_0^T (|z''(L, t)|^2 + |v'_\mathcal{O}(L, t)|^2) \, dt &\leq C \|(f, f_\mathcal{O})\|_{L^1(0, T; H^{-1}(\Omega) \times (L^2(\Omega))^{(m+1)})}^2 \\ \int_0^T (|z'(L, t)|^2 + |v_\mathcal{O}(L, t)|^2) \, dt &\leq C \|(f, f_\mathcal{O})\|_{L^1(0, T; L^2(\Omega)/\mathbf{H} \times ((H_+^1(\Omega))')^{(m+1)})}^2 \end{aligned}$$

for  $(h-N)$ ,  $(c-D)$ , and  $(m-m)$  respectively. In the above  $C = C(T)$  is a generic constant.

(b) Consider

$$\begin{cases} m\ddot{z} - \alpha\ddot{z}'' + Kz'''' = 0 & \text{in } \Omega \times \mathbb{R}^+ \\ \ddot{v}_O - \mathbf{p}_O^{-1}\mathbf{E}_O v_O'' = 0 & \text{in } \Omega \times \mathbb{R}^+ \end{cases} \quad (4.8)$$

with the boundary conditions (1.10) and the initial conditions (1.11). Assume that the initial conditions satisfy  $(z_0, z_1, v_O^0, v_O^1) \in \mathcal{H}$ . Then  $(z, \dot{z}, v_O, \dot{v}_O) \in C([0, T]; \mathcal{H})$  and the solution of (4.8) satisfies for every  $T > \tau$  ( $\tau$  is defined by (1.8)) the following observability and hidden regularity results

$$\begin{aligned} \int_0^T (|z'''(L, t)|^2 + |v_O''(L, t)|^2) dt &\asymp \mathcal{E}_d(0) & (h-N) \\ \int_0^T (|z''(L, t)|^2 + |v_O'(L, t)|^2) dt &\asymp \mathcal{E}_d(0) & (c-D) \\ \int_0^T (|z'(L, t)|^2 + |v_O(L, t)|^2) dt &\asymp \mathcal{E}_{-1}(0) & (m-m) \end{aligned}$$

where  $\mathcal{E}_{-1}$  is defined by (4.5).

**4.2. Observability results for coupled, undamped system.** We now consider the coupled, undamped system, i.e.  $\mathbf{G}_E \neq 0$ ,  $\tilde{\mathbf{G}}_E = 0$ . Consider (1.9) without the damping terms, i.e.,  $\tilde{\mathbf{G}}_E = 0$ :

$$\begin{cases} m\ddot{z} - \alpha\ddot{z}'' + Kz'''' - N^T \mathbf{h}_E \mathbf{G}_E \phi_E' = 0 & \text{on } \Omega \times \mathbb{R}^+ \\ \ddot{v}_O - \mathbf{p}_O^{-1} \mathbf{E}_O v_O'' + \mathbf{p}_O^{-1} \mathbf{h}_O^{-1} \mathbf{B}^T \mathbf{G}_E \phi_E = 0 & \text{on } \Omega \times \mathbb{R}^+ \\ \text{where } (\mathbf{B}v_O = \mathbf{h}_E \phi_E - \mathbf{h}_E N z') \end{cases} \quad (4.9)$$

with the boundary conditions (1.10) and the initial conditions (1.11). Since the generator  $\mathcal{A}_0$  is skew-adjoint, the energy  $\mathcal{E}$  in (2.4) is conserved along solution trajectories.

Now consider the eigenvalue problem corresponding to (4.9)

$$\mathcal{A}_0 \begin{pmatrix} U \\ V \end{pmatrix} = \lambda \begin{pmatrix} U \\ V \end{pmatrix} \Rightarrow V = \lambda U \quad \text{and} \quad \mathcal{A}_1 U = \lambda V. \quad (4.10)$$

Explicitly, (4.10) can be written as

$$\begin{cases} -Ku'''' + N^T \mathbf{h}_E \mathbf{G}_E \phi_E' = \lambda^2 \mathcal{L}u & (4.11a) \\ \mathbf{h}_O \mathbf{E}_O \mathbf{u}'' - \mathbf{B}^T \mathbf{G}_E \phi_E = \lambda^2 \mathbf{p}_O \mathbf{h}_O \mathbf{u}. & (4.11b) \end{cases}$$

The following is the *key uniqueness result* of this paper.

LEMMA 4.1. *The eigenvalue problem (4.11) together with any of the following sets of boundary conditions the boundary conditions*

$$\left\{ \begin{array}{l} u(0, t) = u''(0, t) = u(L, t) = u''(L, t) = u'''(L, t) = 0 \\ \mathbf{u}'(0, t) = \mathbf{u}'(L, t) = \mathbf{u}''(L, t) = 0, \end{array} \right\} \quad (h-N) \quad (4.12)$$

$$\left\{ \begin{array}{l} u(0, t) = u'(0, t) = u(L, t) = u'(L, t) = u''(L, t) = 0 \\ \mathbf{u}(0, t) = \mathbf{u}(L, t) = \mathbf{u}'(L, t) = 0, \end{array} \right\} \quad (c-D), (m-m) \quad (4.13)$$

has only the trivial solution.

**Proof:** We first consider the case of (h-N) boundary conditions. Note that if  $(u, \mathbf{u})$  satisfies (4.11)-(4.12), then  $(z, \mathbf{z}) = (u'', \mathbf{u}'')$  satisfies (4.11) with the boundary conditions

$$\begin{cases} z(0, t) = z''(0, t) = z(L, t) = z'(L, t) = z''(L, t) = 0 & (4.14a) \\ \mathbf{z}'(0, t) = \mathbf{z}(L, t) = \mathbf{z}'(L, t) = 0. & (4.14b) \end{cases}$$

If  $(z, \mathbf{z}) \equiv 0$ , then  $(u'', \mathbf{u}'') \equiv 0$  by using the boundary conditions (4.12). Thus in any of the cases, it is enough to show that (4.11),(4.12) and (4.11),(4.13) have only the trivial solutions.

Now multiply (4.11a) by  $x\bar{u}' - 3\bar{u}$  and multiply (dot product) (4.11b) by  $x\bar{\mathbf{u}}' - 2\bar{\mathbf{u}}$  respectively and add to each other. Then integrating by parts on  $\Omega$  with the use of boundary conditions (4.14) yields :

$$\begin{aligned} 0 &= \int_{\Omega} (-4\lambda^2|u|^2 - 2\alpha\lambda^2|u'|^2 - 3\lambda^2\mathbf{h}_O\mathbf{p}_O\mathbf{u} \cdot \bar{\mathbf{u}} - \mathbf{h}_O\mathbf{E}_O\mathbf{u}' \cdot \bar{\mathbf{u}}') \, dx \\ &+ \int_{\Omega} (-x\lambda^2\bar{u}u' + \alpha x\lambda^2u'\bar{u}'' - Kx\bar{u}''''xu' - \lambda^2\mathbf{h}_O\mathbf{p}_O\mathbf{u}' \cdot x\bar{\mathbf{u}}) \, dx \\ &+ \int_{\Omega} (\mathbf{h}_O\mathbf{E}_O\mathbf{u}' \cdot x\bar{\mathbf{u}}'' - 3\mathbf{G}_E\phi_E \cdot h_E\bar{\phi}_E - \mathbf{G}_E\phi_E' \cdot xh_E\bar{\phi}_E) \, dx. \end{aligned} \quad (4.15)$$

Now we look at the solution  $(\bar{u}, \bar{\mathbf{u}})$  of the eigenvalue problem (4.11) corresponding to the eigenvalue  $\bar{\lambda}$  :

$$\begin{cases} \bar{\lambda}^2\bar{u} - \alpha\bar{\lambda}^2\bar{u}'' + K\bar{u}'''' - N^T\mathbf{h}_E\mathbf{G}_E\bar{\phi}_E' = 0 & (4.16a) \\ \bar{\lambda}^2\mathbf{h}_O\mathbf{p}_O\bar{\mathbf{u}} - \mathbf{h}_O\mathbf{E}_O\bar{\mathbf{u}}'' + \mathbf{B}^T\mathbf{G}_E\bar{\phi}_E = 0. & (4.16b) \end{cases}$$

with the conjugate boundary conditions

$$\begin{cases} \bar{u}(0, t) = \bar{u}''(0, t) = \bar{u}(L, t) = \bar{u}'(L, t) = \bar{u}''(L, t) = 0 & (4.17a) \\ \bar{\mathbf{u}}'(0, t) = \bar{\mathbf{u}}(L, t) = \bar{\mathbf{u}}'(L, t) = 0 & (4.17b) \end{cases}$$

Now multiply (4.16a) by  $xu' + 2u$  and multiply (dot product) (4.16b) by  $x\mathbf{u}' + 3\mathbf{u}$  respectively and add to each other. Then integrating by parts on  $\Omega$  with the use of (4.17) yields

$$\begin{aligned} 0 &= \int_{\Omega} (\bar{\lambda}^2\bar{u}xu' - \alpha\bar{\lambda}^2\bar{u}''xu' + K\bar{u}''''xu' + \bar{\lambda}^2\mathbf{h}_O\mathbf{p}_O\bar{\mathbf{u}} \cdot x\mathbf{u}' - \mathbf{h}_O\mathbf{E}_O\bar{\mathbf{u}}'' \cdot x\mathbf{u}') \, dx \\ &+ \int_{\Omega} (2\bar{\lambda}^2|u|^2 + 2\alpha\bar{\lambda}^2|u'|^2 + 2K|u''|^2 + 3\bar{\lambda}^2\mathbf{h}_O\mathbf{p}_O\bar{\mathbf{u}} \cdot \mathbf{u} + 3\mathbf{h}_O\mathbf{E}_O\bar{\mathbf{u}}' \cdot \mathbf{u}') \, dx \\ &+ \int_{\Omega} (3\mathbf{G}_E\bar{\phi}_E \cdot h_E\phi_E dx + \mathbf{G}_E\bar{\phi}_E \cdot (xh_E\phi_E')) \, dx. \end{aligned} \quad (4.18)$$

Eventually, adding (4.15) and (4.18) gives

$$\begin{aligned}
0 = & \int_{\Omega} (-2(2\lambda^2 - \bar{\lambda}^2)|u|^2 - 2\alpha(\lambda^2 - \bar{\lambda}^2)|u'|^2 + 2K|u''|^2) \, dx \\
& + \int_{\Omega} (-3(\lambda^2 - \bar{\lambda}^2)\mathbf{h}_{\mathcal{O}}\mathbf{p}_{\mathcal{O}}\bar{\mathbf{u}} \cdot \mathbf{u} + 2\mathbf{h}_{\mathcal{O}}\mathbf{E}_{\mathcal{O}}\mathbf{u}' \cdot \bar{\mathbf{u}}') \, dx \\
& + \int_{\Omega} (x(-\lambda^2 + \bar{\lambda}^2)\bar{u}u' + \alpha x(\lambda^2 - \bar{\lambda}^2)u'u'' + (-\lambda^2 + \bar{\lambda}^2)\mathbf{h}_{\mathcal{O}}\mathbf{p}_{\mathcal{O}}\mathbf{u}' \cdot x\bar{\mathbf{u}}) \, dx \\
& + \int_{\Omega} ((\mathbf{G}_E\bar{\phi}_E) \cdot (xh_E\phi'_E) - (\mathbf{G}_E\phi_E) \cdot (xh_E\bar{\phi}'_E)) \, dx. \tag{4.19}
\end{aligned}$$

Note that energy of the undamped system is conserved. Therefore, all eigenvalues are located on the imaginary axis. Now let  $\lambda = \mp is$ ,  $s \in \mathbb{R}^+$ . Then  $\lambda^2$  and  $\bar{\lambda}^2$  have the same sign. Then (4.19) reduces to

$$\begin{aligned}
& \int_{\Omega} 2s^2|u|^2 + 2K|u''|^2 + 2\mathbf{h}_{\mathcal{O}}\mathbf{E}_{\mathcal{O}}\mathbf{u}' \cdot \bar{\mathbf{u}}' \, dx \\
& + \int_{\Omega} ((\mathbf{G}_E\bar{\phi}_E) \cdot (xh_E\phi'_E) - (\mathbf{G}_E\phi_E) \cdot (xh_E\bar{\phi}'_E)) \, dx = 0. \tag{4.20}
\end{aligned}$$

Note that the last two terms in (4.20) are conjugates of each other. Therefore the second integral term is pure imaginary. Hence we have  $u'' = 0$  and  $\mathbf{u}' = 0$ . Using boundary conditions (4.14) we get  $(u, \mathbf{u}) \equiv 0$ . This completes the proof for the (h-N) boundary conditions.

In (c-D) and (m-m) cases, similar calculations again lead to (4.20). Hence using boundary conditions (4.13), we obtain  $(u, \mathbf{u}) \equiv 0$ .  $\square$

The following result is Theorem 6.2 in (Chap VI, [6]), as it applies to our problem.

**THEOREM 4.2.** *Let  $Y = [z, v_{\mathcal{O}}, \dot{z}, \dot{v}_{\mathcal{O}}]^T$  and  $Y_0 = [z^0, v_{\mathcal{O}}^0, z^1, v_{\mathcal{O}}^1]^T$ . Assume the following two conditions.*

(i) *There exists a sufficiently large  $k' \in \mathbb{N}$  such that for  $T > \tau$  ( $\tau$  is defined by (1.8)) we have*

$$\left\{ \begin{aligned} & \int_0^T (|z'''(L, t)|^2 + |v_{\mathcal{O}}''(L, t)|^2) \, dt \asymp \|Y_0\|_{\mathcal{H}}^2 & (h-N) & \tag{4.21a} \\ & \int_0^T (|z''(L, t)|^2 + |v_{\mathcal{O}}'(L, t)|^2) \, dt \asymp \|Y_0\|_{\mathcal{H}}^2 & (c-D) & \tag{4.21b} \\ & \int_0^T (|z'(L, t)|^2 + |v_{\mathcal{O}}(L, t)|^2) \, dt \asymp \|Y_0\|_{\mathcal{H}_{-1}}^2 & (m-m) & \tag{4.21c} \end{aligned} \right.$$

for all solutions of (4.9) with  $Y_0 \in \mathcal{H}_{k'}^{\perp}$  where  $\mathcal{H}_{k'} = \text{span}\{E_{k,l}, 1 \leq k \leq k', 1 \leq l \leq m_k\}$ .

(ii) *There exists  $\bar{T} > 0$  such that for all  $T > \bar{T}$  the estimates (4.21) hold for all solutions of (4.9) with  $Y_0$  such that  $\mathcal{A}Y_0 = \lambda Y_0$ .*

Then for any  $T > \tau$  the estimates (4.21) hold for all solutions  $Y_0 \in \mathcal{H}$  for the (h-N) and (c-D) cases, and  $Y_0 \in \mathcal{H}_{-1}$  for the (m-m) case.



We are now able to prove our main observability result (Theorem 1.2) for the undamped system (with  $\mathbf{G}_E \equiv 0$ ):

LEMMA 4.2. *Let  $T > \tau$ , where  $\tau$  is given by (1.8) and assume that  $\tilde{\mathbf{G}}_E \equiv 0$ . Then solutions of (4.9) satisfy the observability and hidden regularity estimates (1.12).*

**Proof:** This will follow from Theorem 4.2 once we verify the conditions (i) and (ii) of the hypothesis are satisfied.

First we consider the case of (h-N) boundary conditions. Let us write the solution of (4.9) in the form

$$(z, v_{\mathcal{O}})^T = (z_f, v_{\mathcal{O}f})^T + (\hat{z}, \hat{v}_{\mathcal{O}})^T.$$

where  $[z_f, v_{\mathcal{O}f}]^T$  solves (4.6) with

$$(f, f_{\mathcal{O}})^T = [-N^T \mathbf{h}_E \mathbf{G}_E \phi'_E, \mathbf{p}_{\mathcal{O}}^{-1} \mathbf{h}_{\mathcal{O}}^{-1} \mathbf{B}^T \mathbf{G}_E \phi_E]^T$$

and zero initial conditions, and  $(\hat{z}, \hat{v}_{\mathcal{O}})^T$  solves (4.8) with the initial data  $(z^0, v_{\mathcal{O}}^0, z^1, v_{\mathcal{O}}^1)^T$  where  $\mathbf{B}v_{\mathcal{O}} = \mathbf{h}_E \phi_E - \mathbf{h}_E N z'$ . For  $T > \tau$ , we apply part (a) of Theorem 4.1 for  $(z_f, v_{\mathcal{O}f})^T$ , and obtain

$$\begin{aligned} & \int_0^T \left( |z_f'''(L, t)|^2 + |v_{\mathcal{O}f}''(L, t)|^2 \right) dt \\ & \leq \int_0^T \left( \|N^T \mathbf{h}_E \mathbf{G}_E \mathbf{B} v'_{\mathcal{O}}\|_{L^2(\Omega)}^2 + \|N^T \mathbf{h}_E \mathbf{G}_E \mathbf{h}_E N z''\|_{L^2(\Omega)}^2 \right. \\ & \quad \left. + \|\mathbf{p}_{\mathcal{O}}^{-1} \mathbf{h}_{\mathcal{O}}^{-1} \mathbf{B}^T \mathbf{G}_E \mathbf{h}_E^{-1} \mathbf{B} v'_{\mathcal{O}}\|_{(L^2(\Omega))^{m+1}}^2 + \|\mathbf{p}_{\mathcal{O}}^{-1} \mathbf{h}_{\mathcal{O}}^{-1} \mathbf{B}^T \mathbf{G}_E N z''\|_{(L^2(\Omega))^{m+1}}^2 \right) dt \end{aligned}$$

and therefore

$$\int_0^T \left( |z_f'''(L, t)|^2 + |v_{\mathcal{O}f}''(L, t)|^2 \right) dt \leq C_1(\mathbf{G}_E) \int_0^T \left( \|v'_{\mathcal{O}}\|_{(L^2(\Omega))^{m+1}}^2 + \|z''\|_{L^2(\Omega)}^2 \right) dt$$

where  $C_1$  is a function of  $\mathbf{G}_E$ . It follows from (3.6) that

$$\|(z, v_{\mathcal{O}})^T\|_{\mathcal{X}_1}^2 \geq \lambda_1 \|(z, v_{\mathcal{O}})^T\|_{\mathcal{X}}^2, \quad (4.22)$$

where  $\{\lambda_k\}_{k=1}^{\infty}$  are the eigenvalues of the operator  $A_1$ . By equivalence of the energy (see Remark 4.1) and (4.22) it follows that

$$\begin{aligned} & \int_0^T \left( |z_f'''(L, t)|^2 + |v_{\mathcal{O}f}''(L, t)|^2 \right) dt \\ & \leq C_2(\mathbf{G}_E) \int_0^T \left( \frac{1}{\sqrt{\lambda_1}} \|v'_{\mathcal{O}}\|_{(L^2(\Omega))^{m+1}}^2 + \frac{1}{\sqrt{\lambda_1}} \|z''\|_{L^2(\Omega)}^2 \right) dt \leq \frac{C_3(\mathbf{G}_E)}{\sqrt{\lambda_1}} \mathcal{E}_d(0). \end{aligned} \quad (4.23)$$

Now if we use the assumption  $Y_0 \perp \{E_{k,l}, 1 \leq k \leq k', 1 \leq l \leq m_k\}$ , in part (i) of the theorem, then we have

$$\|(z, v_{\mathcal{O}})^T\|_{\mathcal{X}_1}^2 \geq \lambda'_{k'} \|(z, v_{\mathcal{O}})^T\|_{\mathcal{X}}^2 \quad (4.24)$$

and therefore (4.23) can be written as

$$\int_0^T |z_f'''(L, t)|^2 + |v_{\mathcal{O}f}''(L, t)|^2 dt \leq \frac{C_3(\mathbf{G}_E)}{\sqrt{\lambda_{k'}}} \mathcal{E}_d(0). \quad (4.25)$$

Next, for  $T > \tau$  we apply part (b) of Theorem 4.1 together with (4.8) for  $(\hat{z}, \hat{y}_{\mathcal{O}})^T$  respectively, for  $c_1, c_2 > 0$  we get

$$c_1 \mathcal{E}_d(0) \leq \int_0^T |\hat{z}'''(L, t)|^2 + |\hat{v}_{\mathcal{O}}''(L, t)|^2 dt \leq c_2 \mathcal{E}_d(0). \quad (4.26)$$

Since

$$|\hat{z}'''|^2 \leq 2|\hat{z}'''|^2 + 2|z_f'''|^2, \quad |v_{\mathcal{O}}''|^2 \leq 2|\hat{v}_{\mathcal{O}}''|^2 + 2|v_{\mathcal{O}_f}''|^2 \quad (4.27)$$

By combining (4.25), (4.26), and (4.27) we get

$$\int_0^T |z'''(L, t)|^2 + |v_{\mathcal{O}}''(L, t)|^2 dt \leq 2 \left( c_2 + \frac{C_3(\mathbf{G}_E)}{\sqrt{\lambda_{k'}}} \right) \mathcal{E}_d(0). \quad (4.28)$$

Now if we use

$$|\hat{z}'''|^2 \leq 2|z'''|^2 + 2|z_f'''|^2, \quad |\hat{v}_{\mathcal{O}}''|^2 \leq 2|v_{\mathcal{O}}''|^2 + 2|v_{\mathcal{O}_f}''|^2 \quad (4.29)$$

together with (4.25) and (4.26), we obtain

$$\int_0^T (|z'''(L, t)|^2 + |v_{\mathcal{O}}''(L, t)|^2) dt \geq \left( \frac{c_1}{2} - \frac{C_3(\mathbf{G}_E)}{2\sqrt{\lambda_{k'}}} \right) \mathcal{E}_d(0). \quad (4.30)$$

Therefore for  $T > \tau$  inequalities (4.28) and (4.30) give

$$\left( \frac{c_1}{2} - \frac{C_3(\mathbf{G}_E)}{2\sqrt{\lambda_{k'}}} \right) \mathcal{E}_d(0) \leq \int_0^T (|z'''(L, t)|^2 + |v_{\mathcal{O}}''(L, t)|^2) dt \leq 2 \left( c_2 + \frac{C_3(\mathbf{G}_E)}{\sqrt{\lambda_{k'}}} \right) \mathcal{E}_d(0)$$

By choosing  $k'$  large enough as in the assumption together with using (4.4), we obtain

$$\frac{c_1}{2} \mathcal{E}(0) \leq \left( \int_0^T |z'''(L, t)|^2 + |v_{\mathcal{O}}''(L, t)|^2 \right) dt \leq 2c_2 C \mathcal{E}(0).$$

Hence, condition (i) of Theorem 4.2 is fulfilled. Condition (ii) follows from Lemma 4.1.

In the case of (c-D) boundary conditions, (4.24) takes of the following form

$$\|(z, v_{\mathcal{O}})^T\|_{\mathcal{X}_1}^2 \geq \lambda_{k'} \|(z, v_{\mathcal{O}})^T\|_{\mathcal{X}}^2$$

which means

$$\|(z, v_{\mathcal{O}})^T\|_{H_0^2(\Omega) \times (H_0^1(\Omega))^{(m+1)}}^2 \geq \lambda_{k'} \|(z, v_{\mathcal{O}})^T\|_{H_0^1(\Omega) \times (L^2(\Omega))^{(m+1)}}^2.$$

In the case of (m-m) boundary conditions, we use (3.8) so that (4.24) takes of the following form

$$\|(z, v_{\mathcal{O}})^T\|_{H_0^1(\Omega) \times L^2(\Omega)}^2 \geq \lambda_{k'} \|(z, v_{\mathcal{O}})^T\|_{(L^2(\Omega)/\mathbb{H}) \times ((H_{\dagger}^1(\Omega))')^{(m+1)}}^2.$$

The rest of the proof for (c-D) and (m-m) boundary conditions works the same way modulo the obvious modifications.  $\square$

**4.3. Proof of main observability result.** In this subsection we prove our main observability result Theorem 1.2. We show that the general damped system is a bounded perturbation of the undamped system (with  $\tilde{\mathbf{G}}_E = 0$ ) and if  $\|\tilde{\mathbf{G}}_E\|$  is sufficiently small, the observability inequalities (Lemma 4.2) for the undamped case remain valid.

We will need the the following lemma.

**LEMMA 4.3.** *Let  $T > 0$ . For all  $\|\tilde{\mathbf{G}}_E\|$  sufficiently small there exists a constant  $C(\tilde{\mathbf{G}}_E) > 0$  such that for all  $t \in (0, T]$*

$$\begin{cases} C(\tilde{\mathbf{G}}_E) \mathcal{E}(0) \leq \mathcal{E}(T) \leq \mathcal{E}(t) \leq \mathcal{E}(0) & (\text{h-N}), (\text{c-D}) \\ C(\tilde{\mathbf{G}}_E) \mathcal{E}_{-1}(0) \leq \mathcal{E}_{-1}(T) \leq \mathcal{E}_{-1}(t) \leq \mathcal{E}_{-1}(0) & (\text{m-m}), \end{cases} \quad (4.31\text{a})$$

where  $\mathcal{E}$  and  $\mathcal{E}_{-1}$  are defined by (2.4) and (4.5), respectively.

**Proof:** For the (h-N) case, we multiply the first equation in (1.9) by  $\dot{z}''$  and the second equation in (1.9) by  $\dot{v}_{\mathcal{O}}''$ , and integrate by parts in space and time. For the (c-D) and (m-m) cases, we multiply the first equation in (1.9) by  $\dot{z}$ , and the second equation in (1.9) by  $\dot{v}_{\mathcal{O}}$ , and integrate by parts in space and time. We obtain the following energy identities

$$\begin{cases} \mathcal{E}(T) = \mathcal{E}(0) - \int_0^T \left\langle \tilde{\mathbf{G}}_E \dot{\phi}', \mathbf{h}_E^{-1} \dot{\phi}' \right\rangle_{\Omega} dt & (\text{h-N}) \\ \mathcal{E}(T) = \mathcal{E}(0) - \int_0^T \left\langle \tilde{\mathbf{G}}_E \dot{\phi}, \mathbf{h}_E^{-1} \dot{\phi} \right\rangle_{\Omega} dt & (\text{c-D}), (\text{m-m}). \end{cases}$$

Since the dissipation term is bounded in the natural energy space, there exists a constant  $C_1$  such that

$$\begin{cases} \left| - \int_0^T \left\langle \tilde{\mathbf{G}}_E \dot{\phi}', \mathbf{h}_E^{-1} \dot{\phi}' \right\rangle_{\Omega} dt \right| \leq C_1 \|\tilde{\mathbf{G}}_E\| T \mathcal{E}(0) & (\text{h-N}) \\ \left| - \int_0^T \left\langle \tilde{\mathbf{G}}_E \dot{\phi}, \mathbf{h}_E^{-1} \dot{\phi} \right\rangle_{\Omega} dt \right| \leq C_1 \|\tilde{\mathbf{G}}_E\| T \mathcal{E}(0), & (\text{c-D}), (\text{m-m}). \end{cases}$$

Therefore, if  $\|\tilde{\mathbf{G}}_E\|$  is sufficiently small so that  $C(\tilde{\mathbf{G}}_E) := 1 - C_1 \|\tilde{\mathbf{G}}_E\| T > 0$ , i.e.  $\|\tilde{\mathbf{G}}_E\| < \frac{1}{C_1 T}$ , then for each set of boundary conditions

$$C(\tilde{\mathbf{G}}_E) \mathcal{E}(0) \leq \mathcal{E}(T) \leq \mathcal{E}(t) \leq \mathcal{E}(0). \quad (4.34)$$

In particular, (4.31a) holds.

Note that (4.34) implies that if  $\|\tilde{\mathbf{G}}_E\|$  is chosen sufficiently small so that  $C(\tilde{\mathbf{G}}_E) > 0$ , the semigroup  $\{e^{At}\}_{t \geq 0}$  extends to a  $C_0$ -group on  $\mathbb{R}$  for each set of boundary conditions by Proposition 2.7.4 in [19]. This remains true of the semigroup extension defined on  $\mathcal{H}_{-1}$ . In particular, for the case of (m-m) boundary conditions, (4.31b), and hence also the characterization of  $\mathcal{H}_{-1}$  in (3.10) remain valid.  $\square$

Now we can prove our main observability result Theorem 1.2.

**Proof of Theorem 1.2:**

Consider the (h-N) case. We write the solution of (1.9) in the form

$$[z, v_{\mathcal{O}}]^T = [z_f, v_{\mathcal{O}f}]^T + [\hat{z}, \hat{v}_{\mathcal{O}}]^T,$$

where  $[z_f, v_{\mathcal{O}f}]^T$  solves

$$\begin{cases} m\ddot{z} - \alpha\ddot{z}'' + Kz'''' - N^T \mathbf{h}_E \mathbf{G}_E \phi'_E + f(x, t) = 0 & \text{in } \Omega \times \mathbb{R}^+, \\ \ddot{v}_{\mathcal{O}} - \mathbf{p}_{\mathcal{O}}^{-1} \mathbf{E}_{\mathcal{O}} v''_{\mathcal{O}} + \mathbf{p}_{\mathcal{O}}^{-1} \mathbf{h}_{\mathcal{O}}^{-1} \mathbf{B}^T \mathbf{G}_E \phi_E + f_{\mathcal{O}}(x, t) = 0 & \text{on } \Omega \times \mathbb{R}^+, \end{cases} \quad (4.35)$$

with zero initial data and,

$$[f, f_{\mathcal{O}}]^T = [-N^T \mathbf{h}_E \tilde{\mathbf{G}}_E \dot{\phi}'_E, \mathbf{p}_{\mathcal{O}}^{-1} \mathbf{h}_{\mathcal{O}}^{-1} \mathbf{B}^T \tilde{\mathbf{G}}_E \dot{\phi}_E]^T, \quad (4.36)$$

where  $[\hat{z}, \hat{v}_{\mathcal{O}}]^T$  solves (4.9) with the initial data  $[z^0, v_{\mathcal{O}}^0, z^1, v_{\mathcal{O}}^1]^T$ . Since (4.36) is a bounded coupling term in  $\mathcal{H}$ , by equivalence of energy  $\mathcal{E}_d \asymp \mathcal{E}$  (see Remark 4.1), the estimates in part (a) of Theorem 4.1 (which apply to the *decoupled* system) remain valid for (4.35). Thus for any  $T > 0$  we have

$$\begin{aligned} & \int_0^T \left( |z_f'''(L, t)|^2 + |v_{\mathcal{O}f}''(L, t)|^2 \right) dt \\ & \leq \int_0^T \left( \|N^T \mathbf{h}_E \tilde{\mathbf{G}}_E \mathbf{B} \dot{v}'_{\mathcal{O}}\|_{L^2(\Omega)}^2 + \|N^T \mathbf{h}_E \tilde{\mathbf{G}}_E \mathbf{h}_E N \dot{z}''\|_{L^2(\Omega)}^2 \right. \\ & \quad \left. + \|\mathbf{p}_{\mathcal{O}}^{-1} \mathbf{h}_{\mathcal{O}}^{-1} \mathbf{B}^T \tilde{\mathbf{G}}_E \mathbf{h}_E^{-1} \mathbf{B} \dot{v}'_{\mathcal{O}}\|_{(L^2(\Omega))^{m+1}}^2 + \|\mathbf{p}_{\mathcal{O}}^{-1} \mathbf{h}_{\mathcal{O}}^{-1} \mathbf{B}^T \tilde{\mathbf{G}}_E N \dot{z}''\|_{(L^2(\Omega))^{m+1}}^2 \right) dt \\ & \leq C_4(\tilde{\mathbf{G}}_E) \int_0^T \left( \|\dot{z}''\|_{L^2(\Omega)}^2 + \|\dot{v}'_{\mathcal{O}}\|_{(L^2(\Omega))^{m+1}}^2 \right) dt \end{aligned} \quad (4.37)$$

where  $C_4(\tilde{\mathbf{G}}_E) \rightarrow 0$  as  $\|\tilde{\mathbf{G}}_E\| \rightarrow 0$ .

Next, for  $T > \tau$  if we apply part (b) of Theorem 4.1 to  $(\hat{z}, \hat{v}_{\mathcal{O}})^T$ . Hence there exist  $c_1, c_2 > 0$  for which

$$c_1 \mathcal{E}(0) \leq \int_0^T \left( |\hat{z}'''(L, t)|^2 + |\hat{v}_{\mathcal{O}}''(L, t)|^2 \right) dt \leq c_2 \mathcal{E}(0). \quad (4.38)$$

By using (4.27) together with (4.4), (4.31a), (4.37), (4.38) we get

$$\int_0^T \left( |z'''(L, t)|^2 + |v_{\mathcal{O}}''(L, t)|^2 \right) dt \leq 2 \left( c_2 + C_4(\tilde{\mathbf{G}}_E) \right) \mathcal{E}(0).$$

Now by using (4.29) together with (4.31a), (4.37) and (4.38) we get

$$\int_0^T \left( |z'''(L, t)|^2 + |v_{\mathcal{O}}''(L, t)|^2 \right) dt \geq \left( \frac{c_1}{2} - C(\tilde{\mathbf{G}}_E) C_4(\tilde{\mathbf{G}}_E) \right) \mathcal{E}(0).$$

For any fixed  $T > \tau$ , the constant  $C(\tilde{\mathbf{G}}_E)$  is bounded for all sufficiently small  $\|\tilde{\mathbf{G}}_E\|$  (See proof of Lemma 4.3). Hence, for sufficiently small  $\|\tilde{\mathbf{G}}_E\|$ , we get the desired observability result (1.12a).

The rest of the proof for (c-D) and (m-m) boundary conditions works the same way modulo the obvious modifications.  $\square$

**5. Exact controllability results.** Once continuous observability is established on an appropriate function space, exact controllability will also hold on an appropriately defined dual space to the observability space. Here we sketch the procedure for the (h-N) case and indicate the modifications for the (c-D) and (m-m) cases.

**5.1. Proof of Proposition 1.1 and Theorem 1.1 for the (h-N) case.** We first define the transpositional solution of (1.1), (1.2) and (1.5).

By Lemma 2.1,  $\mathcal{A}^* = -\mathcal{A}(-\tilde{\mathbf{G}}_E)$ . Hence the dual backward problem corresponding to (1.1), (1.2) and (1.5) is given by

$$\begin{cases} m\ddot{z} - \alpha\ddot{z}'' + K\dot{z}''' - N^T \mathbf{h}_E \left( \mathbf{G}_E \hat{\phi}_E - \tilde{\mathbf{G}}_E \dot{\hat{\phi}}_E \right)' = 0 & \text{on } \Omega \times \mathbb{R}^+ \\ \mathbf{h}_O \mathbf{p}_O \ddot{v}_O - \mathbf{h}_O \mathbf{E}_O \ddot{v}_O'' + \mathbf{B}^T \left( \mathbf{G}_E \hat{\phi}_E - \tilde{\mathbf{G}}_E \dot{\hat{\phi}}_E \right) = 0 & \text{on } \Omega \times \mathbb{R}^+ \\ \text{where } \mathbf{B} \dot{v}_O = \mathbf{h}_E \hat{\phi}_E - \mathbf{h}_E N \dot{z}' \end{cases} \quad (5.1)$$

with the boundary and terminal conditions

$$\hat{z}(0, t) = \hat{z}''(0, t) = \hat{z}(L, t) = 0, \hat{z}''(L, t) = 0, \quad \hat{v}_O(0, t) = \hat{v}_O'(L, t) = 0 \quad (5.2)$$

$$\hat{z}(x, T_1) = \hat{z}^0(x), \quad \hat{z}'(x, T_1) = \hat{z}^1(x), \quad \hat{v}_O(x, T_1) = \hat{v}_O^0, \quad \hat{v}_O'(x, T_1) = \hat{v}_O^1. \quad (5.3)$$

Now we multiply the first and second equations in (5.1) by  $w''$  and  $y_O''$  respectively where  $(w, y_O)^T$  is the solution of non-homogenous equation (1.1)-(1.5), and then integrate by parts using the boundary conditions (1.2) and (5.2). Combining these (and using the definitions of  $\psi_E$  and  $\hat{\phi}_E$ ) yield

$$\begin{aligned} 0 &= \left[ \int_{\Omega} \left( \dot{z}'' \mathcal{L} w - \dot{z}'' \mathcal{L} \dot{w} + \mathbf{h}_O \mathbf{p}_O \dot{v}_O'' \cdot y_O - \mathbf{h}_O \mathbf{p}_O \dot{v}_O'' \cdot \dot{y}_O + \tilde{\mathbf{G}}_E \dot{\hat{\phi}}_E' \cdot \mathbf{h}_E \psi_E' \right) dx \right]_0^{T_1} \\ &+ \int_0^{T_1} (K \dot{z}'''(L, t) M(t) + \mathbf{h}_O \mathbf{E}_O \ddot{v}_O''(L, t) \cdot \mathbf{g}_O(t)) dt. \end{aligned} \quad (5.4)$$

Now let  $\hat{Y} := (\hat{z}, \hat{v}_O, \dot{\hat{z}}, \dot{\hat{v}}_O)^T$  with  $\hat{Y}(0) = \hat{Y}_0 = (\hat{z}^0, \hat{v}_O^0, \hat{z}^1, \hat{v}_O^1)^T \in \mathcal{H}$ , and let

$$\mathcal{S} = H_0^1(\Omega) \times (L_{\perp}^2(\Omega))^{(m+1)} \times L^2(\Omega) \times ((\tilde{H}^1(\Omega))')^{(m+1)}. \quad (5.5)$$

where  $L_{\perp}^2(\Omega) = \{\varphi \in L^2(\Omega) : \int_{\Omega} \varphi dx = 0\} = (\tilde{L}^2(\Omega))'$ . One can easily prove that the map  $\frac{d^2}{dx^2} : H_{\perp}^2(\Omega) \rightarrow L_{\perp}^2(\Omega)$  is an isomorphism. Moreover, this extends to isomorphism  $\frac{d^2}{dx^2} : H_{\perp}^1(\Omega) \rightarrow (\tilde{H}^1(\Omega))'$ . Consequently,  $\frac{d^2}{dx^2} : \mathcal{H} \rightarrow \mathcal{S}$  is an isomorphism.

Define  $\mathcal{F}_{T_1}$  to be the linear functional on  $\mathcal{H}$  by

$$\begin{aligned} \mathcal{F}_{T_1}(\hat{Y}_0) &= \left\langle (-\mathcal{L} w^1, -\mathbf{h}_O \mathbf{p}_O y_O^1, \mathcal{L} w^0, \mathbf{h}_O \mathbf{p}_O y_O^0), \hat{Y}_0'' \right\rangle_{\mathcal{S}', \mathcal{S}} \\ &- \int_0^{T_1} (K \dot{z}'''(L, t) M(t) + \mathbf{h}_O \mathbf{E}_O \ddot{v}_O''(L, t) \cdot \mathbf{g}_O(t)) dt \\ &+ \left\langle \left( N^T \tilde{\mathbf{G}}_E (\mathbf{h}_E N w^{0''} + \mathbf{B} y_O^{0'}), -\mathbf{B}^T \tilde{\mathbf{G}}_E (N w^{0'} + \mathbf{h}_E^{-1} \mathbf{B} y_O^0), 0, 0 \right), \hat{Y}_0'' \right\rangle_{\mathcal{S}', \mathcal{S}}. \end{aligned} \quad (5.6)$$

Then (5.4) becomes

$$\begin{aligned} \mathcal{F}_{T_1}(\hat{Y}_0) &= \left\langle (-\mathcal{L} \dot{w}, -\mathbf{h}_O \mathbf{p}_O \dot{y}_O, \mathcal{L} w, \mathbf{h}_O \mathbf{p}_O y_O), \hat{Y}'' \right\rangle_{\mathcal{S}', \mathcal{S}} \Big|_{t=T_1} \\ &+ \left\langle \left( N^T \tilde{\mathbf{G}}_E (\mathbf{h}_E N w'' + \mathbf{B} y_O'), -\mathbf{B}^T \tilde{\mathbf{G}}_E (N w' + \mathbf{h}_E^{-1} \mathbf{B} y_O), 0, 0 \right), \hat{Y}'' \right\rangle_{\mathcal{S}', \mathcal{S}} \Big|_{t=T_1}. \end{aligned} \quad (5.7)$$

This identity defines a weak solution of (1.1)-(1.5); more precisely:

**DEFINITION 5.1.** *We say that  $(w, y_{\mathcal{O}}, \dot{w}, \dot{y}_{\mathcal{O}})^T$  is a solution of (1.1)-(1.5) on  $[0, T]$  if  $(w, y_{\mathcal{O}}, \dot{w}, \dot{y}_{\mathcal{O}})^T \in C([0, T], \mathcal{C})$  and (5.7) is satisfied for all  $T_1 \in [0, T]$  and for all  $\hat{Y}_0 \in \mathcal{H}$  where  $\mathcal{C}$  is defined by (1.6).*

To see that Def. 5.1 is fulfilled, first note that by Theorem 1.2,  $(\hat{z}'''(L, \cdot), \hat{v}''_{\mathcal{O}}(L, \cdot)) \in (L^2(0, T))^{(m+2)}$ . Furthermore, since  $\hat{Y}_0 \in \mathcal{H}$ , by Theorem 2.1,  $\hat{Y}''(\cdot, T_1) \in \mathcal{S}$  for all  $T_1 \in [0, T]$ . Therefore, for every  $T_1 \in [0, T]$  the linear form  $\mathcal{F}_{T_1}$  is continuous on  $\mathcal{H}$ . Consequently the duality pairing in (5.7) uniquely defines the  $(-\mathcal{L}\dot{w}, -\mathbf{h}_{\mathcal{O}}\mathbf{p}_{\mathcal{O}}\dot{y}_{\mathcal{O}}, \mathcal{L}w, \mathbf{h}_{\mathcal{O}}\mathbf{p}_{\mathcal{O}}y_{\mathcal{O}})^T \in \mathcal{S}'$  where

$$\mathcal{S}' = H^{-1}(\Omega) \times (\tilde{L}^2(\Omega))^{(m+1)} \times L^2(\Omega) \times (\tilde{H}^1(\Omega))^{(m+1)}.$$

But since

$$\mathcal{L} : H^2(\Omega) \cap H_0^1(\Omega) \rightarrow L^2(\Omega) \quad \text{and} \quad \mathcal{L} : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$$

are isomorphisms it follows that  $(w(\cdot, t), y_{\mathcal{O}}(\cdot, t), \dot{w}(\cdot, t), \dot{y}_{\mathcal{O}}(\cdot, t))^T \in \mathcal{C}$  for all  $t \in \mathbb{R}$ . One can prove the continuity in time, i.e.,  $(w(\cdot, t), y_{\mathcal{O}}(\cdot, t), \dot{w}(\cdot, t), \dot{y}_{\mathcal{O}}(\cdot, t))^T \in C([0, T], \mathcal{C})$  through a standard argument; see e.g., [5, Theorem 2.5]. This proves Proposition 1.1.

Now we prove Theorem 1.1 by the HUM method (i.e. see [11, Chapter 4]). To apply HUM we seek the controls of the form  $(M(t), \mathbf{g}_{\mathcal{O}}) = (\hat{z}'''(L, t), \hat{v}''_{\mathcal{O}}(L, t))$  where  $(\hat{z}, \hat{v}_{\mathcal{O}})$  is the solution of (5.1)-(5.3) for  $T_1 = T$ . By the previous discussion, the backward problem

$$\begin{cases} m\ddot{w} - \alpha\ddot{w}'' + Kw'''' - N^T \mathbf{h}_E (\mathbf{G}_E \psi_E + \tilde{\mathbf{G}}_E \dot{\psi}_E)' = 0 & \text{on } \Omega \times \mathbb{R}^+ \\ \mathbf{h}_{\mathcal{O}} \mathbf{p}_{\mathcal{O}} \dot{y}_{\mathcal{O}} - \mathbf{h}_{\mathcal{O}} \mathbf{E}_{\mathcal{O}} y_{\mathcal{O}}'' + \mathbf{B}^T (\mathbf{G}_E \psi_E + \tilde{\mathbf{G}}_E \dot{\psi}_E) = 0 & \text{on } \Omega \times \mathbb{R}^+ \\ \text{where } \mathbf{B} y_{\mathcal{O}} = \mathbf{h}_E \psi_E - \mathbf{h}_E N w' \end{cases}$$

with boundary and terminal conditions

$$\begin{cases} w(0, t) = w''(0, t) = w(1, t) = 0, & w''(L, t) = \hat{z}'''(L, t) \\ y'_{\mathcal{O}}(0, t) = 0, & y'_{\mathcal{O}}(L, t) = \hat{v}''_{\mathcal{O}}(L, t) \\ w(x, T) = 0, & \dot{w}(x, T) = 0, \quad y_{\mathcal{O}}(x, T) = 0, \quad \dot{y}_{\mathcal{O}}(x, T) = 0 \end{cases}$$

has a unique solution satisfying

$$\begin{aligned} & (-\mathcal{L}\dot{w}(\cdot, 0), -\mathbf{h}_{\mathcal{O}}\mathbf{p}_{\mathcal{O}}\dot{y}_{\mathcal{O}}(\cdot, 0), \mathcal{L}w(\cdot, 0), \mathbf{h}_{\mathcal{O}}\mathbf{p}_{\mathcal{O}}y_{\mathcal{O}}(\cdot, 0))^T \\ & + \left( N^T \tilde{\mathbf{G}}_E (\mathbf{h}_E N w''(\cdot, 0) + \mathbf{B} y'_{\mathcal{O}}(\cdot, 0)), -\mathbf{B}^T \tilde{\mathbf{G}}_E (N w'(\cdot, 0) + \mathbf{h}_E^{-1} \mathbf{B} y_{\mathcal{O}}(\cdot, 0)), 0, 0 \right)^T \in \mathcal{S}'. \end{aligned}$$

Hence, the controllability map  $\Lambda : \mathcal{S} \rightarrow \mathcal{S}'$  defined by

$$\begin{aligned} \Lambda(\hat{Y}_0'') &= (-\mathcal{L}\dot{w}(\cdot, 0), -\mathbf{h}_{\mathcal{O}}\mathbf{p}_{\mathcal{O}}\dot{y}_{\mathcal{O}}(\cdot, 0), \mathcal{L}w(\cdot, 0), \mathbf{h}_{\mathcal{O}}\mathbf{p}_{\mathcal{O}}y_{\mathcal{O}}(\cdot, 0))^T \\ &+ \left( N^T \tilde{\mathbf{G}}_E (\mathbf{h}_E N w''(\cdot, 0) + \mathbf{B} y'_{\mathcal{O}}(\cdot, 0)), -\mathbf{B}^T \tilde{\mathbf{G}}_E (N w'(\cdot, 0) + \mathbf{h}_E^{-1} \mathbf{B} y_{\mathcal{O}}(\cdot, 0)), 0, 0 \right)^T \end{aligned}$$

is continuous from  $\mathcal{S}$  into  $\mathcal{S}'$ . Furthermore, if  $Y_0$  such that

$$(w(\cdot, 0), y_{\mathcal{O}}(\cdot, 0), \dot{w}(\cdot, 0), \dot{y}_{\mathcal{O}}(\cdot, 0))^T = (w^0, v_{\mathcal{O}}^0, w^1, v_{\mathcal{O}}^1)^T,$$

then the control  $(M(t), \mathbf{g}_O) = (\hat{z}'''(L, t), \hat{v}_O''(L, t))$  drives the system (1.1) to rest in time  $T$ . Therefore, Theorem 1.1 is proved if the surjectivity of the map  $\Lambda$  is shown.

Now we choose  $(M(t), \mathbf{g}_O(t)) = (\hat{z}'''(L, t), \hat{v}_O''(L, t))$  in (5.6). Then for  $T > \tau$  and for all  $\hat{Y}_0 \in \mathcal{H}$ , we have

$$\begin{aligned} \left\langle \Lambda(\hat{Y}_0''), \hat{Y}_0'' \right\rangle_{\mathcal{S}', \mathcal{S}} &= \int_0^T (K|\hat{z}'''(L, t)|^2 + \mathbf{h}_O \mathbf{E}_O |\hat{v}_O''(L, t)|^2) dt \\ &\geq c_2 \mathcal{E}(0) \geq c_2 \|\hat{Y}_0''\|_{\mathcal{S}}^2 \end{aligned}$$

where we used (1.12a) with the same constant  $c_2$ . Since  $\Lambda$  is a bounded and coercive, by the Lax-Milgram theorem  $\Lambda$  is surjective. This completes the proof for we complete the proof of Theorem 1.1 for the (h-N) case.

**5.2. Proofs of Proposition 1.1 and Theorem 1.1 for (c-D) and (m-m) cases.** The proofs for (c-D) and (m-m) cases are similar to the proofs for the (h-N) case with several modifications. For example, we multiply the first equation in (5.1) by  $w$  and the second equation in (5.1) by  $y_O$  where  $(w, y_O)^T$  is the solution of non-homogenous equation (1.1)-(1.5), and then integrate by parts using the appropriate boundary conditions. Then, the definition of transpositional solution changes as the following

$$\begin{aligned} \mathcal{F}_T(\hat{Y}_0) &= \left\langle (-\mathcal{L}\dot{w}, -\mathbf{h}_O \mathbf{p}_O \dot{y}_O, \mathcal{L}w, \mathbf{h}_O \mathbf{p}_O y_O), \hat{Y} \right\rangle_{\mathcal{S}', \mathcal{S}} \Big|_{t=T} \\ &+ \left\langle \left( N^T \tilde{\mathbf{G}}_E (\mathbf{h}_E N w'' + \mathbf{B} y_O'), -\mathbf{B}^T \tilde{\mathbf{G}}_E (N w' + \mathbf{h}_E^{-1} \mathbf{B} y_O), 0, 0 \right), \hat{Y} \right\rangle_{\mathcal{S}', \mathcal{S}} \Big|_{t=T}. \end{aligned} \quad (5.8)$$

where the space  $\mathcal{S}$  is defined as the following

$$\mathcal{S} = \begin{cases} \mathcal{H} = H_0^2(\Omega) \times (H_0^1(\Omega))^{(m+1)} \times H_0^1(\Omega) \times (L^2(\Omega))^{(m+1)} & \text{(c-D)} \\ \mathcal{H}_{-1} = H_0^1(\Omega) \times (L^2(\Omega))^{(m+1)} \times (L^2(\Omega)/H) \times ((H_+^1(\Omega))')^{(m+1)} & \text{(m-m).} \end{cases} \quad (5.9a)$$

$$(5.9b)$$

In the above the dual of the space  $L^2(\Omega)/H$  is defined in Lemma 3.2.

Note that (5.8) has  $\hat{Y}$  in the right hand side of the duality pairing whereas  $\hat{Y}''$  appeared in (5.7) for the case of (h-N) boundary conditions. However, the duality pairing between  $\mathcal{S}$  and  $\mathcal{S}'$  is the same. This leads to control spaces  $\mathcal{C}$  defined in (1.6a) and (1.6c) of the same Sobolev order in the cases of (h-N) and (m-m) boundary conditions, as one would expect.

We indicate below other minor modifications needed for (c-D) and (m-m) cases.

**(i) (c-D) case:** In this case the observability result holds on the concrete space  $\mathcal{H} = H_0^2(\Omega) \times (H_0^1(\Omega))^{(m+1)} \times H_0^1(\Omega) \times (L^2(\Omega))^{(m+1)}$ . However, as a consequence of the definition of transpositional solution, the controllability is obtained up to an additive two dimensional space in the velocity component defined in (1.7). To explain this we need the following lemma which is analogous to Lemmata 3.1, 3.2. Proofs can be found in [13] and [14].

**LEMMA 5.1.** (i) The operator  $\mathcal{L}$  is an isomorphism from  $H_0^2(\Omega)$  to  $M^\perp$  where  $M$  is defined by (1.7), (ii)  $(L^2(\Omega)/M)' = M^\perp$ , where the duality is with respect to the  $L^2(\Omega)$  inner product.

By (5.9a) we have  $\mathcal{S}' = H^{-2}(\Omega) \times (H^{-1}(\Omega))^{(m+1)} \times H^{-1}(\Omega) \times (L^2(\Omega))^{(m+1)}$ . We see that  $\mathcal{L}\dot{w}$  is well-defined at any time as an element of  $H^{-2}(\Omega)$  by (5.8). Equivalently,

$\langle \dot{w}, \mathcal{L}\psi \rangle_{L^2(\Omega)}$  is defined for each  $\psi \in H_0^2(0, l)$ . However, the range of  $\mathcal{L}$  on the restricted space  $H_0^2(\Omega)$  is  $M^\perp$  where  $M$  is defined by (1.7). Thus by Lemma 5.1,  $\dot{w}$  is well-defined on the quotient space  $L^2(\Omega)/M$ .

**(ii) (m-m) case:** We find a similar phenomenon in (m-m) case but in the reverse sense: the observability result holds on a factor space  $\mathcal{H}_{-1} = H_0^1(\Omega) \times (L^2(\Omega))^{(m+1)} \times (L^2(\Omega)/H) \times (H_+^1(\Omega))^{(m+1)}$ , while the controllability is obtained on a concrete space defined in (1.6).

By (5.9b) and Lemma 3.2, we have  $\mathcal{S}' = H^{-1}(\Omega) \times (L^2(\Omega))^{(m+1)} \times H^\perp \times (H_+^1(\Omega))^{(m+1)}$ . Therefore,  $\mathcal{L}\dot{w}$  is well-defined since  $\mathcal{L} : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$  is an isomorphism. Equivalently,  $\dot{w} \in H_0^1(\Omega)$  for all  $T \in \mathbb{R}$ . For the well-posedness of  $w$  we investigate the well-posedness of the following term

$$\langle \mathcal{L}w(x, T), \hat{z}(x, T) \rangle_{L^2(\Omega)}. \quad (5.10)$$

By Lemma 3.2, when (5.10) is defined for all  $\hat{z} \in (L^2(\Omega)/H)$ , the term  $\mathcal{L}w(x, T)$  is uniquely defined in  $H^\perp$ . Therefore,  $w$  is uniquely determined as an element in  $H_\#^2(\Omega)$  by Lemma 3.1.  $\square$

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